Real affine varieties and obstruction theories

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Abstract

Let $X = Spec(A)$ be a real smooth affine variety with $\dim X = n \geq 2$, $K = \wedge^n \Omega_{A/\mathbb{R}}$ and L be a rank one projective $A-$module. Let $E(A, L)$ denote the Euler class group and M be the manifold of X. (For this talk we assume M is compact.) Recall that any rank one projective $A-$module L induces a bundle of groups $\mathcal{G}_L$ on M associated to the corresponding line bundle on M. In this talk, we establish a cannonical homomorphism

$$\zeta : E(A, L) \to H_0(M, \mathcal{G}_{LK^*}) \xrightarrow{\sim} H^n(M, \mathcal{G}_{L^*}),$$

where the notation H_0 denotes the 0^{th} homology group and H^n denotes the $n^{th}-$cohomology group with local coefficients in a bundle of groups. In fact, the isomorphism $H_0(M, \mathcal{G}_{LK^*}) \xrightarrow{\sim} H^n(M, \mathcal{G}_{L^*})$ is given by Steenrod's Poincaré duality. Further, we prove that this homomorphism ζ factors through an isomorphism

$$E(\mathbb{R}(X), L \otimes \mathbb{R}(X)) \xrightarrow{\sim} H_0(M, \mathcal{G}_L)$$

where $\mathbb{R}(X) = S^{-1}A$ and S is the multiplicative set of all $f \in A$ that do not vanish at any real point of X.

The obstruction theory in topology is rich and classical. The advent of obstruction theory in algebra (*often known as Euler class theory*) is a more recent phenomenon.

1 Obstruction theory in topology

In topology, for real smooth manifolds M with $\dim(M) = n$ and for vector bundles $\mathcal{E}$ of $rank(\mathcal{E}) = r \leq n$, there is an obstruction group $\mathcal{H}(M, \mathcal{E})$, and an invariant $w(\mathcal{E}) \in \mathcal{H}(M, \mathcal{E})$, to be called the **Whitney class** of $\mathcal{E}$, such that

1. If $rank(\mathcal{E}) = n$ then $\mathcal{E}$ has a nowhere vanishing section if and only if $w(\mathcal{E}) = 0$.
2. Suppose $rank(\mathcal{E}) = r < n$. If $\mathcal{E}$ has a nowhere vanishing section, then $w(\mathcal{E}) = 0$. On the other hand, if $w(\mathcal{E}) = 0$ then $\mathcal{E}|_{M_r}$ has a nowhere vanishing section, where M_r is the $r-$skeleton in any cell complex decomposition of M.
3. In fact,

$$\mathcal{H}(M, \mathcal{E}) = H^r(M, \pi_{r-1}(\mathcal{E}^0))$$

is the cohomology group with local coefficients in the bundle of abelian groups $\pi_{r-1}(\mathcal{E}^0)$ over M

whose fiber at $x \in M$ is the homotopy group $\pi_{r-1}(\mathcal{E}_x \setminus 0)$.

4. Note that, if $\mathcal{E}$ is a Riemannian vector bundle, then

$$\pi_{r-1}(\mathcal{E}^0) = \pi_{r-1}(\mathbb{S}(\mathcal{E})) := \cup_{x\in M}\pi_{r-1}(\mathbb{S}(\mathcal{E})_x)$$

where $\mathbb{S}(\mathcal{E}) \subseteq \mathcal{E}$ is the unit sphere bundle of $\mathcal{E}$. This follows from the fact that $\mathbb{S}(\mathcal{E})$ is a deformation retract of $\mathcal{E}^0$.

5. In ([MaSh3]), a bundle $\mathcal{G}_{\mathcal{L}}$ of abelian group associated to any line bundle $\mathcal{L}$ is defined and an isomorphism $\pi_{r-1}(\mathcal{E}^0) \approx \mathcal{G}_{\wedge^r\mathcal{E}}$, is established. Therefore, given an isomorphism $\chi : \mathcal{L} \xrightarrow{\sim} \wedge^n\mathcal{E}$, there is an isomorphism

$$\varepsilon : H^r(M, \pi_{r-1}(\mathcal{E}_0)) \xrightarrow{\sim} H^r(M, \mathcal{G}_{\mathcal{L}}).$$

By this isomorphism, the Whitney class $w(\mathcal{E})$ can be identified as

$$w(\mathcal{E}, \chi) := \varepsilon(w(\mathcal{E})) \in H^r(M, \mathcal{G}_{\mathcal{L}}).$$

2 Obstruction theory in algebra

In the early nineties, Nori outlined a program for an obstruction theory in algebra. The program of Nori mirrors the already existing theory in topology for vector bundles $\mathcal{E}$ over a manifold M with $\dim M = rank(\mathcal{E}) = n \geq 2$. Accordingly, for smooth affine algebras A over infinite fields k with $\dim(A) = n \geq 2$, and for projective $A-$modules L with $rank(L) = 1$, Nori outlined a definition ([MS], later generalized in [BS2]) of an obstruction group $E(A, L)$, which contains an invariant $e(P, \chi)$ for any projective $A-$module P of rank n with orientation $\chi : L \xrightarrow{\sim} \wedge^n P$, such that conjecturally, $e(P, \chi) = 0$ if and only if $P \approx Q \oplus A$ for some projective $A-$module Q. We denote,

$$e(P) := e(P, \mathrm{id}) \in E(A, \wedge^n P).$$

In fact, an orientation $\chi : L \xrightarrow{\sim} \wedge^n P$ induces an isomorphism

$$f_\chi : E(A, \wedge^n P) \xrightarrow{\sim} E(A, L) \qquad \textit{and} \qquad f_\chi(e(P)) = e(P, \chi).$$

Essentially, all the conjectures given at the time when the program was outlined were proved and the program of Nori flourished beyond all expectations. Among the major and important papers on this program are ([Ma1, MS, MV, BS1, BS2, BS4, BDM]).

It is also worth mentioning, that the program of Nori was preceded by the work of M. P. Murthy and N. Mohan Kumar ([MkM, Mk2, Mu1, MM]) on similar obstructions for projective modules P over (smooth) affine algebras A over algebraically closed fields k, with $\dim A = rank(P)$, where the top Chern class $C_0(P)$ was used as obstruction.

The Main point

The main point of this talk is to reconcile the recently developed theory in algebra for projective modules of top rank and the existing classical theory in topology, in such top rank case.

Notation 2.1 Let $X = Spec(A)$ be a real smooth affine variety. We assume $\dim A = n \geq 2$. We fix a few notations as follows:

1. The manifold of real points of X will be denoted by $M = M(X)$. We assume $M \neq \phi$ and is compact. We have $\dim M = n$.
2. The ring of all real-valued continuous functions on M will be denoted by $C(M)$.
3. We write $\mathbb{R}(X) = S^{-1}A$, where S denotes the multiplicative set of all functions $f \in A$ that do not vanish at any real point of X.
4. Usually, L will denote a projective $A-$module of rank one and $\mathcal{L}$ will denote the line bundle over M, whose module of cross sections comes from L, i.e. $\Gamma(\mathcal{L}) = L \otimes C(M)$ ([Sw]).

Our main question is whether there is a canonical homomorphism from the algebraic obstruction group

$E(A, L)$ to the topological obstruction group $\mathcal{H}(M, \mathcal{L})$.

3 The Main Results

In this section, we describe results from our recent papers ([MaSh1, MaSh2, MaSh3]). Our final theorem is the following:

Theorem 3.1 With notations as in 2.1, we establish an isomorphism

$$\zeta : E(\mathbb{R}(X), L) \xrightarrow{\sim} H^n(M, \mathcal{G}_{\mathcal{L}^*}).$$

Note that this gives a completely algebraic description of the singular cohomomolgy $H^n(M, \mathbb{Z})$.

Further, for a projective $\mathbb{R}(X)-$module P with $rank(P) = n$, and $L = \wedge^n P$, we have

$$\zeta(e(P)) = w(\mathcal{E}^*).$$

3.1 The oriented case

In this section, we outline the proof of the main theorem 3.1, in the simpler case of oriented varieties. We bring the following definition from topology.

Definition 3.2 Let M be a smooth oriented manifold of dimension n. Let $v \in B$ be a point and V be an open neighborhood of v. Suppose

$$f = (f_1, \ldots, f_n) : V \to \mathbb{R}^n$$

is an ordered $n-$tuple of smooth functions such that f has an isolated zero at v. Now fix a parametrization $\varphi : \mathbb{R}^n \xrightarrow{\sim} U \subseteq V$, compatible with the orientation of M, where U is a neighbourhood of $v = \varphi(0)$. By modifying φ, we assume that $f\varphi$ vanishes only at the origin $0 \in \mathbb{R}^n$. Define index $j_v(f_1, \ldots, f_n)$ to be the degree of the map

$$\gamma = \frac{f\varphi}{\| f\varphi \|} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}.$$

That means,

$$j_v(f_1, \ldots, f_n) = \deg(\gamma) := H_n(\gamma)(1) \in \mathbb{Z}$$

where

$$H_n(\gamma) : H_n(\mathbb{S}^n, \mathbb{Z}) = \mathbb{Z} \to H_n(\mathbb{S}^n, \mathbb{Z}) = \mathbb{Z}$$

is the induced homomorphism.

Recall, that $X = spec(A)$ is said to be oriented, if $K_X = \wedge^n \Omega_{A/\mathbb{R}}$ is trivial.

Proof of theorem 3.1 in the oriented case: In the oriented case, with M compact, the obstruction group in topology

$$\mathcal{H}(M, \mathcal{G}_{M\times\mathbb{R}}) = H^n(M, \mathbb{Z}) \approx \mathbb{Z}$$

is the more familiar singular cohomology groups with $\mathbb{Z}$ coefficients. Therefore, in this case, we want to define a homomorphism (isomorphism):

$$\zeta : E\left(\mathbb{R}(X), \mathbb{R}(X)\right) \xrightarrow{\sim} H^n(M, \mathbb{Z}) = \mathbb{Z}.$$

1. Recall, generators of $E\left(\mathbb{R}(X), \mathbb{R}(X)\right)$ are given by (m, ω), where m is a (real) maximal ideal of $\mathbb{R}(X)$ and $\omega : \mathbb{R}(X)/m \xrightarrow{\sim} \wedge^n m/m^2$ is an isomorphism, to be called a local orientation.
2. Such a generator, gives rise to an equivalence class of generators $(f_1, f_2, \ldots, f_n)$ of m/m^2.
3. Now we define

   $$\zeta : E(\mathbb{R}(X), \mathbb{R}(X)) \to H^n(M, \mathbb{Z}) \approx \mathbb{Z}.$$

   by

   $$\zeta(m, \omega) = j_v(f_1, f_2, \ldots, f_n)$$

   where m and $(f_1, f_2, \ldots, f_n)$ are as above.

4. The relations in $E(\mathbb{R}(X), \mathbb{R}(X))$ are induced by complete intersections

$$(f_1, f_2, \dots, f_n) = m_1 \cap m_2 \cap \cdots \cap m_k$$

where $m_1, \dots, m_k$ are distinct (real) maximal ideals of $\mathbb{R}(X)$. Also let $v_i \in M$ denote the point in corresponding to m_i.

To see that ζ is well defined, we only need to see that

$$\sum j_{v_i}(f_1, f_2, \dots, f_n) = e(T)$$

is the topological Euler class of the trivial bundle T, and hence $e(T) = 0$.

5. Now let P be projective $\mathbb{R}(X)-$module of rank n, with trivial depterminant. Then, for an orientation $\chi : \mathbb{R}(X) \xrightarrow{\sim} \wedge^n P$ the Euler class $e(P, \chi)$ is induced by surjective homomorphisms

$$s : P \twoheadrightarrow m_1 \cap m_2 \cap \cdots \cap m_k$$

where $m_1, \dots, m_k$ are distinct (real) maximal ideals of $\mathbb{R}(X)$. Again, with $v_i \in M$ as the points corresponding to m_i, we have

$$\zeta(e(P, \chi)) = \sum j_{v_i}(f_{1i}, \dots, f_{ni})$$

where $(f_{1i}, \dots, f_{ni})$ is a set of generators of m_i/m_i^2, induced by s, respecting the local orientation χ.

From results in topology, the right hand side is the Whitney class $w(\mathcal{E}^*)$. This completes the proof. ∎

4 Applications

We have the following consequence of the main theorem 3.1.

Theorem 4.1 Let $X = Spec(A)$, $\mathbb{R}(X), M \neq \phi$ be as in (2.1) and let $CH_0(\mathbb{R}(X))$ denote the Chow group of zero cycles of $\mathbb{R}(X)$ modulo rational equivalence. Then, the following diagram

$$\begin{array}{ccc} E(\mathbb{R}(X), L) & \xrightarrow{\zeta} & H^n(M, \mathcal{G}_{\mathcal{L}}) \\ \Theta \downarrow & & \downarrow \mu \\ CH_0(\mathbb{R}(X)) & \xrightarrow{\zeta_0} & H^n(M, \mathbb{Z}/(2)). \end{array}$$

commutes, where Θ, μ are the natural homomorphisms and ζ_0 is an isomorphism.

The following is another application:

Theorem 4.2 We use the notations as in (2.1), Then, there is a canonical isomorphism $\zeta_0 : CH_0(\mathbb{R}(X)) \xrightarrow{\sim} H^n(M, \mathbb{Z}/(2))$ such that the diagram

$$\begin{tikzcd}
K_0(A) \arrow[r] \arrow[d, "C_0"'] & K_0(\mathbb{R}(X)) \arrow[r] \arrow[d, "C_0"'] & KO(M) \arrow[d, "sw_n"] \\
CH_0(A) \arrow[r] & CH_0(\mathbb{R}(X)) \arrow[r, "\zeta_0"] & H^n(M, \mathbb{Z}/(2))
\end{tikzcd}$$

commutes. Here C_0 denotes the top Chern class homomorphism, and sw_n denotes the top Stiefel-Whitney class homomorphism.

Example 4.3 As in (2.1), suppose $X = Spec(A)$ is a smooth real affine variety with $\dim X = n \geq 2$ and M is the manifold of real points of X. We assume M is nonempty.

Assume $CH^{\mathbb{C}}(X) \neq 0$, where $CH^{\mathbb{C}}(X) \subseteq CH_0(X)$ denotes the subgroup the Chow group of zero cycles of X modulo rational equivalence. Let L be a projective $A-$module with $rank\ (L) = 1$.

Then, there is a projective $A-$module with $rank\ (P) = n$ and $\det P \xrightarrow{\sim} L$ such that

1. P does not have a unimodular element,
2. but the corresponding vector bundle $V = V(P)$, whose module of sections $\Gamma(V) = P \otimes C(M)$ has a nowhere vanishing section.

Proof. The following diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & E^{\mathbb{C}}(L) \arrow[r] \arrow[d, "\varphi", "\wr"'] & E(A, L) \arrow[r] \arrow[d, "\Theta"] & E(\mathbb{R}(X), L) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & CH^{\mathbb{C}}(X) \arrow[r] & CH_0(A) \arrow[r] & CH_0(\mathbb{R}(X)) \arrow[r] & 0 \end{tikzcd}$$

Let x be a complex point in X such the cycle $[x] \in CH_0(X)$ is nonzero. Let m be a maximal ideal corresponding to x.

By [Mu1, DM2], there is a projective $A-$module of rank n and $\det P = L$ and an orientation $\chi : L \xrightarrow{\sim} \wedge^n P$

such that the top chern class $C_0(P) = x \neq 0$. The construction assures that $e(P,\chi) = \varphi^{-1}(x) \in E^{\mathbb{C}}(L)$. Since $e(P,\chi) \neq 0$ and so P does not have a unimodular element. Therefore, $e(P \otimes \mathbb{R}(X), \chi \otimes \mathbb{R}(X)) = 0$. Now, suppose $V = V(P)$ is the vector bundle on M with module of sections $\Gamma(M,V) = P \otimes C(M)$. Then, by our theorem, $e(V^*, \chi') = \zeta(e(P \otimes \mathbb{R}(X), \chi \otimes \mathbb{R}(X))) = 0$. So, V^* has a nowhere vanishing section and so does V. This completes the proof. ∎

Example 4.4 M. P. Murthy communicated the following example:

Let

$$A = \frac{\mathbb{R}[X_0, X_1, \ldots, X_n]}{\left(\sum X_i^d - 1\right)} \qquad \textit{where} \quad d > n+1.$$

Then $CH^{\mathbb{C}}(A) \neq 0$.

5 Real spheres

In this section, we do some calculus of Euler class (obstruction) theory on the algebraic spheres. Write

$$A_n = \frac{\mathbb{R}[X_0, X_1, \ldots, X_n]}{(X_0^2 + X_1^2 + \cdots + X_n^2 - 1)}.$$

Swan ([Sw2]) established that the algebraic and topological $K-$theory of spheres are the same. So, by Bott periodicity theorem ([ABS]) the following chart

$n =$	$8r$	$8r+1$	$8r+2$	$8r+3$	$8r+4$	$8r+5$	$8r+6$	$8r+7$
$\widetilde{K_0}(A_n) \approx \widetilde{KO}(\mathbb{S}^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

describes the Grothendieck groups $K_0(A_n)$ of A_n. We also have the Chow group of zero cycles

$$CH_0(A_n) = \mathbb{Z}/(2)$$

and by results in ([BDM]), the Euler class group

$$E(A_n, A_n) = E\left(\mathbb{R}(\mathbb{S}^n), \mathbb{R}(\mathbb{S}^n)\right) = \mathbb{Z}.$$

Let T_n denote the tangent bundle on $Spec(A_n)$. We compute the Euler class

$$e(T_n, \chi) = \pm 2 \quad for\, n\, even; \quad and \quad e(T_n, \chi) = 0 \quad for\, n\, odd.$$

This provides a fully algebraic proof that the algebraic tangent bundles T_n over even dimensional spheres $Spec(A_n)$

do not have a free direct summand. Note that the Euler class $e(T_n, \chi)$ agrees with the topological Euler class $e(T_n, \chi)$ (see [MiS]).

The other problem we consider is whether every element in $E(A_n, A_n)) = \mathbb{Z}$ can be realized as the Euler class $e(P, \chi)$ of a projective $A_n$module P with $rank(P) = n$ and orientation $\chi : A \xrightarrow{\sim} \wedge^n P$?

The following summary evolved:

Theorem 5.1 *Let P denote any projective A_n-module of rank $n \geq 2$ and $\chi : A_n \xrightarrow{\sim} \det P$ be an orientation. Then*

1. *For $n = 8r+3, 8r+5, 8r+7$ we have $\tilde{K_0}(A_n) = 0$. So, the top Chern class $C_0(P) = 0$ and $e(P, \chi) = 0$.*

2. *For $n = 8r+6$, we have $\tilde{K_0}(A_n) = 0$. So, $C_0(P) = 0$, and hence $e(P, \chi)$ is always even. Further, for any even integer N there is a projective A_n-module Q with $rank(Q) = n$ and an orientation $\eta : A_n \xrightarrow{\sim} \det Q$, such that $e(Q, \eta) = N$.*

3. *For $n = 8r+1$, we have $\tilde{K_0}(A_n) = \mathbb{Z}/2$. If $e(P, \chi)$ is even then $C_0(P) = 0$. So, $P \approx Q \oplus A_n$ and $e(P, \chi) = 0$ for all orientations χ. So, only even value $e(P, \chi)$ can assume is zero.*

4. Now consider the remaining cases, $n = 8r, 8r + 2, 8r + 4$. *We have*

$$\tilde{K}_0(A_{8r}) = \mathbb{Z}, \quad \tilde{K}_0(A_{8r+2}) = \mathbb{Z}/2, \quad \tilde{K}_0(A_{8r+4}) = \mathbb{Z}.$$

As in the case of $n = 8r + 6$, *for any even integer* N, *for some* (Q, η) *the Euler class* $e(Q, \eta) = N$. *If* N *is odd, then* $e(P, \chi) = N$ *for some* (P, χ) *if and only if* $e(P, \chi) = 1$ *for some* (P, χ) *if and only if the top Stiefel-Whitney class* $w_n(V) = 1$ *for some vector bundle* V *with* $rank(V) = n$.

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