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Cancellation and complete intersections

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This talk is on the joint work with Bangere Purnaprajna.

An important open problem in the area of complete intersections is whether local complete intersection surfaces and higher dimensional varieties in affine n-spaces are set theoretic complete intersection. For the case of curves, this is a theorem due to Mohan Kumar.

Here we will talk about height 2 ideals. We first state the following theorem of Bloch-Murthy-Szpiro ([BMS]).

Theorem 1.1 Let $R=k[X_1,X_2,X_3,X_4]$ be a polynomial ring over an algebraically closed field k and I be an ideal of height 2 in R and $A=R/I,$ such that X = $Spec(R/I)$ is smooth over k. Let ${}_I={}^2(I/I^2)^{-1}$ and K_X = $C_1({}_I)$ be the canonical divisor of X. If $rK_X{}^2 =0$ in the Chow group $CH^2(X)$, for some integer $r>0$, then X is set-theoretically complete intersection.

Following theorem is helpful to understand the statement of this theorem *K-theoretically.*

Theorem 3.2 Let X =$Spec(A)$ be a smooth affine algebra over an algebraically closed field $k.$ Assume that projective *A-modules* of rank 2 have cancellation property. Let L be a projective *A-module* of rank one. Then the following conditions are equivalent:

1. L is generated by 2 elements.
2. $C_1(L)^2 =0$ in $CH^2(X)$.
3. $L \quad L^{-1} \quad A^2.$

We will give a 5-dimentional version of the theorem 1.1 assuming that projective modules of rank 2 over 3-folds have cancellation property.

For (an increasing sequence of) positive integers n, Mohan Kumar ([MK]) constructed smooth affine algebras A with $dim\ A = n$ and stably free projective *A-modules* of rank n-2 that are not free. He also posed the following question.

LRC-problem: Let A be a (smooth) affine algebra of dimension n over an algebraically closed field k. Does the projective *A-modules* of rank n-1 have the cancellation property? That is, for a projective *A-module* P with $rank(P)=n-1$, does $P \quad A \quad Q \quad A \quad P \quad Q?$

If LRC-problem has an affirmative answer for 3-folds, then we prove a three dimensional analogue of the above theorem of Bloch-Murthy-Szpiro as follows:

Theorem 3.4 Suppose k is an algebraically closed field and $R=k[X_1,X_2, ..., X_5]$ is a polynomial ring. Let I be a locally complete intersection ideal of R with $height(I) = 2$ and $A=R/I$ is smooth over $k.$ Assume that all rank 2 projective *A-modules* have cancellation property.

Suppose $rK^2 = 0$ for some $r > 0$ where $K=C_1(\ {}_I).$ Then I is set-theoretic complete intersection.

Before we go into our main results we state the following (monic polynomial version of) theorem of Ferrand and Szpiro.

Lemma 2.1 Let *R*= *A[X]* be a polynomial ring over a noetherian commutative ring *A* and *I* be a locally complete intersection ideal of height 2 in *R*. Assume that *I* contains a monic polynomial and there is a surjective map $I/I^2 \ \ {}_I = Ext^2(R/I,R)$. Then *I* is set theoretically generated by 2 elements.

Proof.

The argument of Ferrand-Szpiro goes as follows:

Let ***J*** be defined by the exact sequence

$$0 \ \ J/I^2 \ \ I/I^2 \ \ {}_I \ \ 0$$

Then ***J*** is a locally complete intersection ideal with ***rad(J) =rad(I)*** and $\ {}_J$ = Ext^2(R/J,R) = Ext^1(J,R) is one generated .

The generator ***e*** of Ext^1(J,R) correspond to an exact sequence

$$0 \ \ R \ \ P \ \ J \ \ 0$$

It follows that *P* is a projective *R-module* of rank 2. Since *J* contains a monic polynomial, we have $P_f \ \ {}_f^2$. Hence by the theorem of Quillen-Suslin, it follows that *P* is free and hence *J* is complete intersection ideal.

Exercise. Why *P* is projective?

So, the **main Idea** is to get a map like in this lemma.

2 Main Results

We now prove a lemma which is a three dimensional analogue of a result in [BMS].

Lemma 2.2 Let *R=R'[X]* be polynomial ring over a noetherian commutative ring *R'* with *dim R' =4* (so *dim R=5*). Let *I* be a locally complete intersection ideal of height 2 in *R* that contains a monic polynomial and *A= R/I*. Assume that all rank 2 projective *A=R/I* modules have cancellation property. Then

1. $I/I^2 \ \ A^{\ -1}$
2. Consider the following conditions
   a. is generated by two elements.
   b. $I/I^2 \ \ ^{-2}$
   c. $^{-2}$ is generated by 2 elements.

   Then *a) b) c).*

Proof.

Here we have *dim A =3.* We have an exact sequence

$$0 \ \ P \ \ R^n \ \ I \ \ 0$$

where *P* is a projective R-module of rank *n-1.* Since we can assume that *rank P = n-1 > dim R* and because *I* has a monic polynomial *f* it follows that $P_f \ \ {}_f^{n-1}$.

Hence, from theorem of Quillen and Suslin it follows P^{n-1}.

Tensoring the above sequence with A we get the following exact sequence:

$$\boldsymbol{0 \;\; L \;\; A^{n-1} \;\; A^{n} \;\; I/I^{2} \;\; 0}$$

So, $[I/I^2] = [L]+[A]$ in $K_0(A)$. Since rank 2 projective *dim A-modules* have cancellation property, we have $I/I^2 \;\; L \;\; A$. We also have ${}^{2}\, I/I^2 = L = \;\; {}^{-1}$. So, $I/I^2\; A \;\; {}^{-1}$.

Now we prove the second statement:

a) b): By 1) we have $I/I^2 \;\; {}^{-1} \;\; A$ and also $A^2 \;\;\; {}^{-1}$. So,

$$\boldsymbol{I/I^{2} \;\; A \;\; (A\; {}^{-1}) \;\; A \;\; {}^{-1} \;\; A^{2} \;\; {}^{-1} \;\; (\; {}^{-1})}$$
$$\boldsymbol{(\; {}^{-1} \;\; A^{2}) \;\;\; {}^{-1} \;\; (\; {}^{-1}) \;\;\; A\; {}^{-2}.}$$

Since all rank 2 projective *A-modules* have cancellation property, we have $I/I^2 \;\; {}^{-2}$.

b) c): By 1) and b) it follows that

$$\boldsymbol{A\; {}^{-1} \;\;\; {}^{-2}.}$$

By dualizing we get

$$\boldsymbol{A \;\;\; {}^{-1\; 2}.}$$

Now adding A to both sides and using 1) we have:

$$\boldsymbol{A^{2} \;\;\; (A\; {}^{-1})\; {}^{2}\; I/I^{2\; 2}\; (\; {}^{-2})\; {}^{2}}$$

Since we have assumed that rank 2 projective *A-modules* have cancellation property, by canceling we get $A^2 \;\; {}^{-2\; 2}$. So, ^{2} is generated by 2 elements.

Following theorem relates the cancellation property of rank 2 projective modules with set theoretic complete intersection property of height 2 locally complete intersection ideals.

Theorem 2.1 Let $R=R'[X]$ be a polynomial ring over a noetherian commutative ring R' with $\dim R' = 4$ (so $\dim R=5$). Let $I \;\; R$ be a locally complete intersection ideal of height 2 that contains a monic polynomial and $A = R/I$. Assume that projective *A-modules* of rank 2 have the cancellation property. Let $\;\; = Ext^2(A,R)$. Assume that ^{r} is generated by 2 elements, for some integer $r \geq 1$. Then I is a set-theoretic complete intersection ideal.

Proof.

By the above theorem we have, $I/I^2\; A\; {}^{-1}$. So, we have an exact sequence

$$\boldsymbol{0 \;\; A \;\; I/I^{2} \;\; {}^{-1} \;\; 0,}$$

where the map $A \;\; I/I^2$ is given by an element $f \;\; I$. So, $I=Rf + I'$, where $I^2 \;\; I'\; I$ is an ideal and $I'/I^2 \;\; {}^{-1}$. Write $J=I^r+Rf$. We will see that J is locally complete intersection ideal. For $p \;\; Spec(R)$ containing I , let $I_p = (f,g)$, where image of g generate ${}^{-1}$. So, $J_p= (f,g)^r + R_pf$. Hence $J_p= (g^r,f)$, is complete intersection.

Let L be the cokernel of the map given by f as follows

$$\boldsymbol{f\!: R/J \;\; J/J^{2} \;\; L \;\; 0.}$$

Since, locally J_p is generated by (f,g^r), as in the above paragraph, we have L is locally generated by one element. So, L is a line bundle.

Since $rad(I) = rad(J)$, all rank 2 projective R/J-*modules* also have cancellation property. By the above lemma, we have $J/J^2 \; R/J \; {}_J^{-1}$ and $L \; {}_J^{-1}$. It follows that

1. $J/J^2 \; (R/J)f \; {}_J^{-1}$.
2. ${}_J^{-1} \; J/(J^2+Rf)$.

Consider the exact sequence

$$\mathbf{0 \; R/J \; J/J^2 \; {}_J^{-1}.}$$

By tensoring the sequence with R/I, we get

$$\mathbf{0 \; R/I \; J/IJ \; {}_J^{-1} \; R/I \; 0}$$

So,

$$\mathbf{{}_J^{-1} \; R/I \; J/(IJ,f) \; (I^r,f)/(I^{r+1},f) \; {}_I^{-r},}$$

which is 2 generated. So, ${}_J^{-1}$ is also 2 generated. Hence $A^2 \; {}_J^{-1} \; {}_J$. By lemma 2.2 *2a) 2b),* it follows that there is a surjective map

$$\mathbf{J/J^2 \; {}_J.}$$

Now it follows from lemma 2.1 that J is set theoretically generated by 2 elements. This completes the proof of this theorem.

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3 Smooth 3-folds in A^5

Recall that for any smooth affine variety X over an algebraically closed field k, the Chow group $CH_0(X) = CH^n(X)$ is torsion free (see [Sr] and [Mu]).

Notation 3.1 Let A be a smooth affine algebra over an algebraically closed field k and $X=Spec(A)$.

1. Recall $CH^r(X)$ denotes the Chow group of cycles of codimension r.
2. For a projective A-*module* P, $C_i(P) \; CH^i(X)$ will denote the *i-th* Chern class of P.

Following is a restatement of the theorem of Murthy and Mohan Kumar ([MKM]) in our context of LRC-question.

Theorem 3.1 Let $X = Spec(A)$ be a smooth affine 3-fold over an algebraically closed field k. Assume that projective A-*modules* of all rank (*equivalently of rank 2*) have cancellation property. Then, for any two projective modules P and Q of same rank, $P \; Q$ if and only if $C_i(P) = C_i(Q)$ for $i=1,2,3$.

Proof.

() This implication is obvious.

() The theorem of Mohan Kumar and Murthy ([MKM]) implies that P and Q are stably isomorphic. Now $P \; Q$ by the cancellation property.

Theorem 3.2 Let X =*Spec(A)* be a smooth affine algebra over an algebraically closed field k. Assume that projective *A-modules* of rank 2 have cancellation property. Let L be a projective *A-module* of rank one. Then the following conditions are equivalent:

1. L is generated by 2 elements.
2. $C_1(L)^2 = 0$ in $A^2(X)$.
3. $L \ \ L^{-1} \ \ A^2$.

Proof.

1) 2): Since L is two generated, we have $L \ \ L^{-1} \ \ A^2$. So, $C(L \ \ L^{-1}) = 1$. Hence $(1+C^{-1}(L))(1-C^{-1}(L)) = 1$ and $C^{-1}(L)^2 = 0$.

2) 3): It follows that the total Chern class $C(L \ \ L^{-1}) = 1$. Hence $[L] \ \ [L^{-1}] = [A^2]$ in $K_0(A)$ and by cancellation $L \ \ L^{-1} \ \ A^2$.

3) 1): This implication is obvious.

Theorem 3.3 Suppose k is an algebraically closed field and $R=R'[X]$ a polynomial ring over an affine algebra R' with $\dim R' = 4$. Let I be a locally complete intersection ideal in R of height 2 that contains a monic polynomial. Assume $A = R/I$ is smooth and rank 2 projective *A-modules* have cancellation property. Then the following conditions are equivalent:

1. $K^2 = 0$ in $A^2(X)$ (where $K = C_1(_I)$).
2. $_I$ is two generated.

In all these cases, we have $I/I^2 \ \ _I \, _I^{-2}$ and that I is a set-theoretic complete intersection ideal.

Proof.

Obvious from above.

Theorem 3.4 Suppose k is an algebraically closed field and $R=R'[X]$ is a polynomial ring over an affine algebra R' with $\dim R' = 4$. Let I be a locally complete intersection ideal of R with $height(I) = 2$ and $A=R/I$ is smooth over k. Assume that all rank 2 projective *A-modules* have cancellation property. Suppose $rK^2 = 0$ for some $r > 0$ where $K=C_1(_I)$. Then I is set-theoretic complete intersection.

Proof.

We can assume that $r^2K^2=0$. Let $L = {}^r$. Then the total Chern classes

$$\boldsymbol{C(L \ \ L^{-1}) = (1+rK)(1 - rK) = 1.}$$

So, $L \ \ L^{-1} \ \ A^2$. So, $_I{}^r$ is two generated. By Theorem 2.1, I is a set-theoretic complete intersection.

Following corollary follows from the above theorem.

Corollary 3.1 Let k be an algebraically closed field and $R=k[X_1, \dots ,X_5]$ be polynomial ring and : R A =A'[X] be a surjective map onto a polynomial algebra over a smooth algebra A' of dimension 2. Let I be a the kernel of and $K=C_1 (_I)$. If $rK^2 = 0$ for some positive integer r then I is set theoretic complete intersection.

Proof.

Note that rank 2 projective *A-modules* have cancellation property. Now the corollary follows from the above theorem.

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