Chapter 3: Second Order ODE §3.2 Homogeneous Linear SODEs with constant coefficients

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Goals

- In this section, we start working with Homogeneous LSODEs, with constant coefficients. Such an equation is written as:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{where} \quad a, b, c \in \mathbb{R} \qquad (1)$$

$$\text{also written as} \quad ay'' + by' + c = 0$$

The Characteristic equation

- Magically, solutions of (1) would be exponential functions $y = e^{rt}$, for some values of r; checked as folows.
- Substituting $y = e^{rt}$ in (1) we get

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = (ar^2 + br + c)e^{rt} = 0$$

- It follows, $y = e^{rt}$ is a solution of (1) if and only if

$$ar^2 + br + c = 0 \tag{2}$$

This (2) is called the characteristic equation (CE) of (1).

Three cases of the roots of CE

- The roots of the CE (2) are: $r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. We have three situations:
  - The CE (2) has two distinct real roots $r = r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$. (*To be dealt with in* this section).
  - The CE (2) has two equal real roots $r = r_1 = r_2 \in \mathbb{R}$. (*To be dealt with in* §3.5).
  - The CE (2) has two complex roots $r = r_1, r_2 \in \mathbb{C}$, with $r_1 = \overline{r_2}$. (*To be dealt with in* §3.4).

The case of two real roots $r_1 \neq r_2$

Assume $r_1 \neq r_2$ are two distinct roots of the CE (2).

- Then $\begin{cases} y_1(t) = e^{r_1 t} \\ y_2(t) = e^{r_2 t} \end{cases}$ are two solutions of (1).
  It follows, for any two arbitrary constants c_1, c_2,

  $$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \qquad (3)$$

  is a solution of the LSODE (1). This can be seen by direct checking or by a "slick" method:

- Write

$$\mathcal{L}(y) = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy \qquad \text{Then}$$

$$\mathcal{L}(c_1y_1 + c_2y_2) = c_1\mathcal{L}(y_1) + c_2\mathcal{L}(y_2) = c_1 * 0 + c_2 * 0 = 0$$

- Note, for any funtion $z = \varphi(t)$, we can define

$$\mathcal{L}(z) = a\frac{d^2z}{dt^2} + b\frac{dz}{dt} + cz \quad \text{and for} \quad z_1 = \varphi_1(t), z_2 = \varphi_2(t)$$

$$\mathcal{L}(c_1z_1 + c_2z_2) = c_1\mathcal{L}(z_1) + c_2\mathcal{L}(z_2)$$

This is the "linearity" property of the "operator" $\mathcal{L}$.

Continued

- (3) will be called the general solution of the LSODE (1).

IVP: The Particular solution

- As in FODE, we need some additional information to determine the constants c_1, c_2 in (3). Since we have two unknown constants, we would need two conditions.
- Along with LSODE (1),
  the initial value problem (IVP) has the form

$$\begin{cases} a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = & 0 \\ y(t_0) = & y_0 \\ y'(t_0) = & y'_0 \end{cases} \qquad (4)$$

Continued

- We use the general solution (3) $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$.
- Differentiating, $y'(t) = c_1r_1e^{r_1t} + c_2r_2e^{r_2t}$
- Now, use the initial values:

$$\begin{cases} c_1e^{r_1t_0} + c_2e^{r_2t_0} = & y_0 \\ c_1r_1e^{r_1t_0} + c_2r_2e^{r_2t_0} = & y_0' \end{cases} \tag{5}$$

- Solving (5), we get c_1, c_2. Combining with (3), we obtain particular solution.
- Using Cramer's rule, a formula for c_1, c_2 can be given.

Example 1

- Find the general solution of homogeneous SODE $2\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$
- The CE: $2r^2 + r - 1 = 0$. So, $r_1 = \frac{1}{2}$, $r_2 = -1$
- By (3) the general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{\frac{t}{2}} + c_2 e^{-t}$$

Example 2

- Find the general solution of homogeneous SODE $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0$
- The CE: $2r^2 + 3r = 0$. So, $r_1 = 0$, $r_2 = -\frac{3}{2}$.
- By (3) the general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 + c_2 e^{-\frac{3t}{2}}$$

Example 3

- Find the general solution of homogeneous SODE $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$
- The CE: $r^2 - r - 6 = 0$. So, $r_1 = 3$, $r_2 = -2$
- By (3) the general solution is

$$y = c_1e^{r_1t} + c_2e^{r_2t} = c_1e^{3t} + c_2e^{-2t}$$

Example 4

- Find the general solution of homogeneous SODE
$\frac{d^2y}{dx^2} - (1+\pi)\frac{dy}{dx} + \pi y = 0$
- The CE: $r^2 - (1+\pi)r + \pi = 0$. So, $r_1 = 1$, $r_2 = \pi$
- By (3) the general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{t} + c_2 e^{\pi t}$$

Example 5

- Solve the initial value problem:

$$\left\{\begin{array}{rl} 3\frac{d^2y}{dt^2}-4\frac{dy}{dt}+y= & 0\\ y(0)= & 4\\ y'(0)= & 0\end{array}\right\}, \quad \text{Find} \quad \lim_{t\to\infty} y(t).$$

  and sketch the graph.
- The CE: $3r^2-4r+1=0$. So, $r_1=1,\ r_2=\frac{1}{3}$
- By (3) the general solution is

$$y=c_1e^{r_1t}+c_2e^{r_2t}=c_1e^t+c_2e^{\frac{t}{3}} \tag{6}$$

Continued

- Differentiate: $y' = c_1 e^t + \frac{1}{3} c_2 e^{\frac{t}{3}}$.
- USE initial values:

$$\begin{cases} c_1 e^0 + c_2 e^0 = & 4 \\ c_1 e^0 + \frac{1}{3} c_2 e^0 = & 0 \end{cases} \Longrightarrow \begin{cases} c_1 + c_2 = & 4 \\ c_1 + \frac{1}{3} c_2 = & 0 \end{cases}$$

$$\Longrightarrow \begin{cases} c_1 + c_2 = & 4 \\ 3c_1 + c_2 = & 0 \end{cases} \Longrightarrow \quad c_1 = -2, c_2 = 6$$

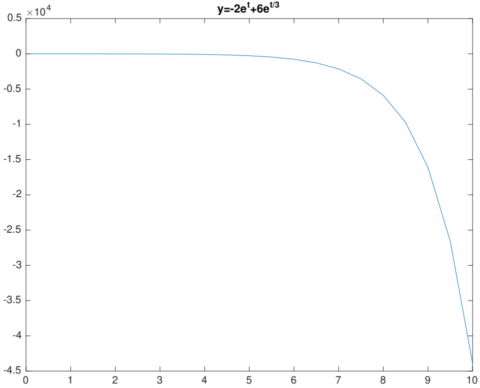
- Particular Solution: By 6,

$$y = c_1 e^t + c_2 e^{\frac{t}{3}} = -2e^t + 6e^{\frac{t}{3}}$$

Continued

- To compute the limit, write $z = e^{\frac{t}{3}}$.
  Then,
  $$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(-2e^{t} + 6e^{\frac{t}{3}}\right)$$
  $$= \lim_{z\to\infty} \left(-2z^{3} + 6z\right) = -\infty$$
  (*Recall, the higher order term dominates, while computing such limits.*)

The graph of the solution:



Matlab commands to Plot Graphs

To plot the graph, following commands were given:

- t=[0:.5:10];
- y=-2*exp(t)+6*exp(t/3);
- plot(t,y), title('The Equation')

Example 6

- Solve the initial value problem:

$$\left\{\begin{array}{rl} \frac{d^2y}{dt^2}+4\frac{dy}{dt}-12y= & 0\\ y(1)= & 1\\ y'(1)= & 1\end{array}\right\}, \quad \text{Find} \quad \lim_{t\to\infty} y(t).$$

  and sketch the graph.
- The CE: $r^2+4r-12=0$. So, $r_1=-6$, $r_2=2$
- By (3) the general solution is

$$y=c_1e^{r_1t}+c_2e^{r_2t}=c_1e^{-6t}+c_2e^{2t} \qquad (7)$$

Continued

- Differentiate: $y' = -6c_1e^{-6t} + 2c_2e^{2t}$.
- USE initial values:

$$\left\{\begin{array}{rl} c_1e^{-6} + c_2e^2 = & 1 \\ -6c_1e^{-6} + 2c_2e^2 = & 1 \end{array}\right. \Longrightarrow \left\{\begin{array}{rl} c_1 + c_2e^8 = & e^6 \\ -6c_1 + 2c_2e^8 = & e^6 \end{array}\right.$$

$$\Longrightarrow \left\{\begin{array}{rl} 6c_1 + 6c_2e^8 = & 6e^6 \\ -6c_1 + 2c_2e^8 = & e^6 \end{array}\right. \Longrightarrow c_1 = \frac{e^6}{8}, c_2 = \frac{7}{8e^2}$$

- Particular Solution: By 7,

$$y = c_1e^{-6t} + c_2e^{2t} = \frac{e^6}{8}e^{-6t} + \frac{7}{8e^2}e^{2t}$$

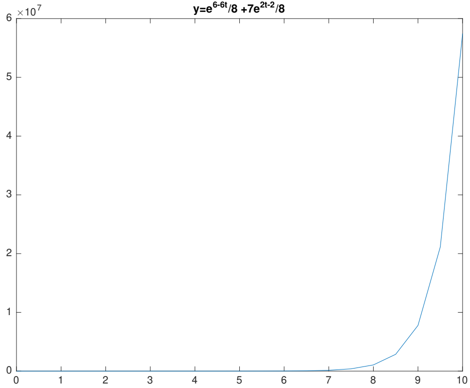
Continued

- To compute the limit, write $z = e^{2t}$.
  Then,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(\frac{e^6}{8}e^{-6t} + \frac{7}{8e^2}e^{2t}\right)$$

$$= \lim_{z\to\infty}\left(\frac{e^6}{8}z^{-3} + \frac{7}{8e^2}z\right) = \infty$$

The graph of the solution:



Matlab commands to Plot Graphs

To plot the graph, following commands were given:

- t=[0:.5:10];
- y=exp(6-6*t)/8 +7*exp(2*t-2)/8;
- plot(t,y), title('The Equation')

Example 7

- Solve the initial value problem:

$$\left\{\begin{array}{rcl} \frac{d^2y}{dt^2} - 2\frac{dy}{dt} & = & 0 \\ y(0) & = & \alpha \\ y'(0) & = & 2 \end{array}\right\}, \text{ For what value of } \alpha, \quad \lim_{t\to\infty} y(t)$$

  is finite?
- The CE: $r^2 - 2r = 0$. So, $r_1 = 0$, $r_1 = 2$
- By (3) the general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 + c_2 e^{2t} \qquad (8)$$

Continued

- Differentiate: $y' = 2c_2e^{2t}$.
- USE initial values:

$$\begin{cases} c_1 + c_2 = & \alpha \\ 2c_2 = & 2 \end{cases} \implies \begin{cases} c_1 = & \alpha - 1 \\ c_2 = & 1 \end{cases}$$

- Particular Solution: By 8,

$$y = c_1 + c_2e^{2t} = (\alpha - 1) + e^{2t}$$

Continued

- Now
$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left((\alpha - 1) + e^{2t}\right) = \infty$$
So, the limit is never finite, irrespective of the value of α.

Example 7a

- Solve the initial value problem:

$$\left\{\begin{array}{rcl} \frac{d^2y}{dt^2} - 2\frac{dy}{dt} & = & 0 \\ y(0) & = & 0 \\ y'(0) & = & \alpha \end{array}\right\}, \text{ For what value of } \alpha, \lim_{t\to\infty} y(t)$$

  is finite?
- The ODE is same as in Example 7. The general solution is, as in (8).

Continued

- Differentiate: $y' = 2c_2e^{2t}$.
- USE initial values:

$$\left\{\begin{array}{rl} c_1 + c_2 = & 0 \\ 2c_2 = & \alpha \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{rl} c_1 = & -\frac{\alpha}{2} \\ c_2 = & \frac{\alpha}{2} \end{array}\right.$$

- Particular Solution: By 8,

$$y = c_1 + c_2e^{2t} = -\frac{\alpha}{2} + \frac{\alpha}{2}e^{2t}$$

Continued

- Now

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(-\frac{\alpha}{2}+\frac{\alpha}{2}e^{2t}\right) = -\frac{\alpha}{2}+\frac{\alpha}{2}\lim_{t\to\infty} e^{2t}$$

So, the limit is finite, only when $\alpha = 0$.

Example 7b

- Solve the initial value problem:

$$\left\{\begin{array}{rcl} \frac{d^2y}{dt^2}+2\frac{dy}{dt} & = & 0 \\ y(0) & = & 0 \\ y'(0) & = & \alpha \end{array}\right\}, \text{ For what value of } \alpha, \ \lim_{t\to\infty} y(t)$$

  is finite?
- The CE: $r^2+2r=0$. So, $r_1=0$, $r_1=-2$
- By (3) the general solution is

$$y = c_1e^{r_1t}+c_2e^{r_2t} = c_1+c_2e^{-2t} \tag{9}$$

Continued

- Differentiate: $y' = -2c_2e^{-2t}$.
- USE initial values:

$$\begin{cases} c_1 + c_2 = & 0 \\ -2c_2 = & \alpha \end{cases} \implies \begin{cases} c_1 = & \frac{\alpha}{2} \\ c_2 = & -\frac{\alpha}{2} \end{cases}$$

- Particular Solution: By 9,

$$y = c_1 + c_2e^{-2t} = \frac{\alpha}{2} - \frac{\alpha}{2}e^{-2t}$$

Continued

- Now
$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(\frac{\alpha}{2} - \frac{\alpha}{2}e^{-2t}\right) = \frac{\alpha}{2}$$
So, the limit is always finite.

Example 8

- Solve the initial value problem:

$$\left\{\begin{array}{rl} \frac{d^2y}{dt^2} - 4y = & 0 \\ y(0) = & -1 \\ y'(0) = & 2 \end{array}\right\}, \text{ Also, compute } \lim_{t\to\infty} y(t).$$

- The CE: $r^2 - 4 = 0$. So, $r_1 = -2$, $r_2 = 2$
- By (3) the general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{-2t} + c_2 e^{2t} \tag{10}$$

Continued

- Differentiate: $y' = -2c_2e^{-2t} + 2c_2e^{2t}$.
- USE initial values:

$$\begin{cases} c_1 + c_2 = & -1 \\ -2c_1 + 2c_2 = & 2 \end{cases} \implies \begin{cases} c_1 + c_2 = & -1 \\ -c_1 + c_2 = & 1 \end{cases}$$

So, $c_1 = -1$ and $c_2 = 0$

- Particular Solution: By 10,

$$y = c_1e^{-2t} + c_2e^{2t} = -e^{-2t}$$

Continued

- Now
$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(-e^{-2t}\right) = 0$$