Chapter 5
§5.2 B System of 1^{st} linear ODE
Determinant of a Matrix

Satya Mandal

U. Kansas

Arrowtic *K*-Theory

Fall 2025

Goals

- We will define determinant of SQUARE matrices, inductively, using the definition of Minors and cofactors.
- We will see that determinant of triangular matrices is the product of its diagonal elements.

Overview of the definition

- Given a square matrix A, the determinant of A will be defined as a scalar, to be denoted by $det(A)$ or $|A|$.
- We define determinant inductively. That means, we first define determinant of 1×1 and 2×2 matrices. Use this to define determinant of 3×3 matrices. Then, use this to define determinant of 4×4 matrices and so.

Determinant of 1×1 and 2×2 matrices

- For a 1×1 matrix $A = [a]$ define $det(A) = |A| = a$.
- Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad define \quad det(A) = |A| = ad - bc.$$

Example 1

Let

$$A = \begin{pmatrix} 2 & 17 \\ 3 & -2 \end{pmatrix} \quad then \quad det(A) = |A| = 2*(-2)-17*3 = -53$$

Example 2

Let

$$A = \begin{pmatrix} 3 & 27 \\ 1 & 9 \end{pmatrix} \quad then \quad det(A) = |A| = 3 * 9 - 1 * 27 = 0.$$

Minors of 3×3 matrices

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then, the **Minor** M_{ij} of a_{ij} is defined to be the determinant of the 2×2 matrix obtained by deleting the i^{th} row and j^{th} column.

For example

$$M_{22} = \begin{vmatrix} a_{11} & & a_{13} \\ & & \\ a_{31} & & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Like wise

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}.$$

Cofactors of 3×3 matrices

Let A the 3×3 matrix as in the above frame.
Then the **Cofactor** C_{ij} of a_{ij} is defined,
by some sign adjustment of the minors, as follows:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$So, \left\{ \begin{array}{l} C_{11} = (-1)^{1+1} M_{11} = M_{11} = a_{22}a_{33} - a_{23}a_{33} \\ C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -(a_{11}a_{32} - a_{12}a_{31}) \\ C_{32} = (-1)^{3+2} M_{32} = -(a_{11}a_{23} - a_{13}a_{21}). \end{array} \right.$$

Determinant of 3×3 matrices

Let A be the 3×3 matrix as above.
Then the **determinant** of A is defined by

$$\det(A) = |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

This definition may be called
"definition by expansion by cofactors, along the first row". It is possible to define the same by expansion by 2^{nd}, or 3^{rd} row.
It can be dome by expansion by any columns,
by improvisation of the same formula.

Example 3

Let

$$A = \begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 0 \\ -2 & 1 & 1 \end{vmatrix}$$

Compute the minor M_{11}, M_{12}, M_{13}, the cofactors C_{11}, C_{12}, C_{13} and the determinant of A.

Solution:

Then minors

$$M_{11} = \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix}, M_{12} = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix}, M_{13} = \begin{vmatrix} 3 & -2 \\ -2 & 1 \end{vmatrix}$$

Or

$$M_{11} = -2, \quad M_{12} = 3, \quad M_{13} = -1$$

Continued

So, the cofactors

$$C_{11} = (-1)^{1+1} M_{11} = -2, \quad C_{12} = (-1)^{1+2} M_{12} = -3,$$

$$C_{13} = (-1)^{1+3} M_{13} = -1$$

So,

$$|A| = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} = 2*(-2) + 1*(-3) + 1*(-1) = -8$$

Inductive process of definition

- We defined determinant of size 3×3, using the determinant of 2×2 matrices.
- Now we can do the same for 4×4 matrices. This means first define minors, which would be determinant of 3×3 matrices. Then define Cofactors by adjusting the sign of the Minors.Then, use the cofactors fo define the determinant of the 4×4 matrix.
- Then we can define minors, cofactors and determinant of 5×5 matrices. The process continues.

Minors of $n \times n$ Matrices

We assume that we know how to define determiant of $(n-1) \times (n-1)$ matrices. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{13} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

be a square matrix of size $n \times n$. The **minor** M_{ij} of a_{ij} is defined to be the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column.

Cofactors and Determinant of $n \times n$ Matrices

Let A be a $n \times n$ matrix.

- Define
$$C_{ij} = (-1)^{i+j} M_{ij} \quad \textit{which iscalled the } \textbf{cofactor} \textit{ of } a_{ij}.$$
- Define
$$det(A) = |A| = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$$
This would be called a definiton by expasion by cofactors, along first row.

Alternative Method for 3×3 matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Form a new 3×5 matrix by adding 1^{st}, 2^{nd} column to A:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & 31 & a_{32} \end{matrix}$$

Continued

Then $|A|$ can be computed as follows:

- add the product of all three entries in the three left to right diagonals.
- add the product of all three entries in the three right to left diagonals.
- Then, $|A|$ is the difference.

Definition.

Definitions. Let A be a $n \times n$ matrix.

- We say A is Upper Triangular matrix, all entries of A below the main diagonal (left to right) are zero. In notations, if $a_{ij} = 0$ for all $i > j$.
- We say A is Lower Triangular matrix, all entries of A above he main diagonal (left to right) are zero. In notations, if $a_{ij} = 0$ for all $i < j$.

Theorem

Theorem Let A be a triangular matrix of order n. Then $|A|$ is product of the main-diagonal entries. Notationally,

$$|A| = a_{11}a_{22}\cdots a_{nn}.$$

Proof. The proof is easy when $n = 1, 2$. We prove it when $n = 3$. Let use assume A is lower triangular. So,

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Continued

We expand by the first row:

$$|A| = a_{11}C_{11} + 0C_{12} + 0C_{13} = a_{11}C_{11}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

For upper triangular matrices, we can prove similarly, by column expansion. For higher order matrices, we can use mathematical induction. ■

Example

Compute the determiant, by expansion by cofactors, of

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{pmatrix}$$

Solution.

- The cofactors

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 4 \\ 0 & 2 \end{vmatrix} = 8, C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = 2$$

- $$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix} = -4$$
- So, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} =$

$$2 * 8 + (-1) * 2 + 3 * (-4) = 2$$

Example

$$\textit{Let} \quad A = \begin{pmatrix} 3 & 7 & -3 & 13 \\ 0 & -7 & 2 & 17 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix} \quad \textit{Compute } det(A).$$

Solution. This is an upper triangular matrix. So, $|A|$ is the product of the diagonal entries. So

$$|A| = 3 * (-7) * 4 * 5 = -420.$$

Example

$$\textit{Solve} \quad \begin{vmatrix} x+3 & 1 \\ -4 & x-1 \end{vmatrix} = 0$$

Solution. So,

$$(x+3)(x-1) - 1 * (-4) = 0 \quad or \quad x^2 + 2x + 1 = 0$$

$$(x+1)^2 = 0 \quad or \quad x = -1.$$