Chapter 3
Second Order ODE
§3.5 Complex roots of the CE

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Homogeneous LSODEs

- Recall the two forms of Homogeneous LSODEs:

$$\begin{cases} \mathcal{L}(y) = y'' + p(t)y' + q(t)y = 0 \\ \mathcal{L}(y) = P(t)y'' + Q(t)y' + R(t)y = 0 \end{cases} \tag{1}$$

where $p(t), q(t), P(t), Q(t), R(t)$ are functions of t.

- **The Trivial Solution:** $y = 0$ is a solution for any homogeneous linear equation (1)

Complex solutions to real

Sometimes the equation (1) would have complex solutions, while we are interested only in real solution.
The following theorem helps.

Theorem 3.5.1: Consider the homogeneous equation (1), where $p(t), q(t)$ are real valued functions of t.
Let $y = \varphi(t) = u(t) + iv(t)$
be a complex solution of the ODE (1), where
$u(t)$, $v(t)$ are the real part the imaginary part of y.
Then, both $y = u(t)$, $y = v(t)$ are solutions of (1).
Proof: Use linearity.

Complex roots of the CE

Consider a 2^{nd}-Order Homogeneous linear ODE, with constant coefficients:

$$\mathcal{L}(y) = ay" + by' + cy = 0 \qquad a, b, c \in \mathbb{R} \tag{2}$$

$$\text{The CE of (2) is :} \quad ar^2 + br + c = 0 \tag{3}$$

- In §3.2, we dealt with the situations, when (3) had two unequal real roots.
- In this section, we deal with the case, when the CE (3) would have complex roots.

Continued

- When the CE (3) has complex roots, $b^2 - 4ac < 0$. The roots are

$$\begin{cases} r_1 = \lambda + i\mu = \frac{-b+\sqrt{b^2-4ac}}{2a} \\ r_2 = \lambda - i\mu = \frac{-b-\sqrt{b^2-4ac}}{2a} \end{cases} \quad \text{where} \quad i = \sqrt{-1}$$

  We say, r_1 and r_2 are conjugate of each other.

- As in §3.2 (2) has two solutions:

$$\begin{cases} y_1(t) = e^{r_1 t} = \exp[(\lambda + i\mu)t] = e^{\lambda t} e^{i\mu t} \\ y_2(t) = e^{r_2 t} = \exp](\lambda - i\mu)t] = e^{\lambda t} e^{-i\mu t} \end{cases} \tag{4}$$

- However, (4) involves complex exponentiation $e^{i\mu t}, e^{-i\mu t}$.

Complex Exponentiation

- For real numbers θ, we define $e^{i\theta} = \cos\theta + i\sin\theta$.
- For complex numbers $z = \rho + i\theta$ define

$$e^{z} := e^{\rho+i\theta} := e^{\rho}e^{i\theta} = e^{\rho}(\cos\theta + i\sin\theta)$$

- All the rules of exponentiation that you are familiar with work, with this definition of e^z. In particular

$$e^{z+w} = e^{z}e^{w} \qquad \text{for all} \quad z, w \in \mathbb{C}.$$

- *Justifications for defining complex exponentiation e^z this way, is dealt with in the Complex Analysis Courses.*

Solution of (2)

- So the solution (4) of (2) reduces to:

$$\begin{cases} y_1(t) = e^{\lambda t}(\cos \mu t + i \sin \mu t) \\ y_2(t) = e^{\lambda t}(\cos \mu t - i \sin \mu t) \end{cases} \tag{5}$$

- By Theorem 3.5.1 both the real and complex parts (of y_1 or y_2) are solution of (2). We get two real solutions:

$$\begin{cases} u(t) = e^{\lambda t} \cos \mu t \\ v(t) = e^{\lambda t} \sin \mu t \end{cases} \tag{6}$$

Continued

- Define the Wronskian of *u*and *v* (*of any two functions*) as

$$W(u,v)(t) = \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix}$$

$$= \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - e^{\lambda t}\mu\sin\mu t & \lambda e^{\lambda t}\sin\mu t + e^{\lambda t}\mu\cos\mu t \end{vmatrix}$$

$$= e^{2\lambda t}\begin{vmatrix} \cos\mu t & \sin\mu t \\ \lambda\cos\mu t - \mu\sin\mu t & \lambda\sin\mu t + \mu\cos\mu t \end{vmatrix} = \mu e^{2\lambda t}$$

Continued

- So, (since $\mu \neq 0$), Wronskian $W(u,v)(t) = \mu e^{2\lambda t} \neq 0$.
- So, u, v form a fundamental set of solutions of (2).
- So, the general (real) solution of (2) has the form
$$y = c_1 u(t) + c_2 v(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t \qquad (7)$$
  where c_1, c_2 are arbitrary constants.
- We can write the same as
$$y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) \qquad (8)$$

Behavior of the solution

If the CE (3) has complex roots or $\mu \neq 0$, then, the solution (8) has two factors:

- The exponential factor:

$$E(t) = e^{\lambda t}$$

Depending on the sign of λ this part will "blow up" to ∞ or "decay" to the $x-$axis (horizontal asymptote).

- The periodic factor.

$$\Phi(t) = c_1 \cos \mu t + c_2 \sin \mu t \quad \text{with periodicity} = \frac{2\pi}{\mu}$$

- Further,

$$-(|c_1| + |c_2|) \leq \Phi(t) \leq (|c_1| + |c_2|)$$

  The graph of $\Phi(t)$ contributes to a steady oscillation.

- The behavior of the solution y (as in (8)) will be a combination of
  (1) the exponential rise/decay due to $E(t)$ and
  (2) the periodic oscillation due to $\Phi(t)$.

Continued

So, the nature of the solutions is summarized as follows:

- If $\lambda = 0$ then the solution would be a steady oscillation.
- If $\lambda > 0$, it will be unsteady oscillation.
- If $\lambda < 0$ the oscillation will stabilize with time.

Example 1

Consider the IVP:

$$\begin{cases} y'' - 4y' + 8y = 0 \\ y(\pi/8) = 0 \\ y'(\pi/8) = e^{\pi/4} \end{cases}$$

- Solve the problem
- Sketch the graph
- Describe the nature of the solution, as $t \to \infty$

Solution

- The CE: $r^2 - 4r + 8 = 0$
- Roots of the CE: $r_1 = 2 + 2i$, $r_2 = 2 - 2i$.
- By solution (8), the general solution

$$y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) = e^{2t}(c_1 \cos 2t + c_2 \sin 2t)$$

- The answer to the last part: the solution would be an unsteady oscillation; because the exponential part is e^{2t}

Continued

- Before we use the initial values, compute

$$y' = 2e^{2t}(c_1 \cos 2t + c_2 \sin 2t) + e^{2t}(-2c_1 \sin 2t + 2c_2 \cos 2t)$$

- Initial value conditions:

$$\begin{cases} y(\pi/8) = 2e^{\pi/4}\left(\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right) = 0 \\ y'(\pi/8) = 2e^{\pi/4}\left(\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right) + e^{\pi/4}\left(-\frac{2c_1}{\sqrt{2}} + \frac{2c_2}{\sqrt{2}}\right) = e^{\pi/4} \end{cases}$$

$$\begin{cases} c_1 + c_2 = 0 \\ \frac{4}{\sqrt{2}}c_2 = 1 \end{cases} \implies \begin{cases} c_1 = -\frac{1}{2\sqrt{2}} \\ c_2 = \frac{1}{2\sqrt{2}} \end{cases}$$

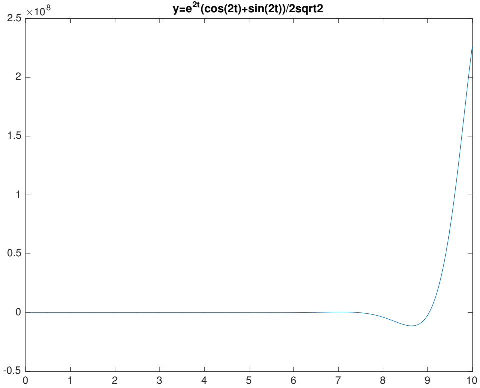
Continued

- So, the solution is

$$y = e^{2t}\left(-\frac{1}{2\sqrt{2}}\cos 2t + \frac{1}{2\sqrt{2}}\sin 2t\right)$$

- Repeat: $y = y(t)$ has an unsteady/unstable oscillation.

Graph of $y = y(t)$:



Example 2 (Dampened Oscillation)

Consider the IVP:

$$\begin{cases} y'' + 4y' + 5y = 0 \\ y(\pi/4) = 2 \\ y'(\pi/4) = -4 \end{cases}$$

- Solve the problem
- Sketch the graph
- Describe the nature of the solution, as $t \to \infty$

Solution

- The CE: $r^2 + 4r + 5 = 0$
- Roots of the CE: $r_1 = -2 + i$, $r_2 = -2 - i$.
- By solution (8), the general solution

$$y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) = e^{-2t}(c_1 \cos t + c_2 \sin t)$$

- The answer to the last part: the solution will be an stabilized/dampened/ decaying oscillation; because the exponential part is e^{-2t}

Continued

- Before we use the initial values, compute

$$y' = -2e^{-2t}(c_1 \cos t + c_2 \sin t) + e^{-2t}(-c_1 \sin t + c_2 \cos t)$$

- Initial value conditions:

$$\begin{cases} y(\pi/4) = e^{-\pi/2}\left(\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right) = 2 \\ y'(\pi/4) = -2e^{-\pi/2}\left(\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right) + e^{-\pi/2}\left(-\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right) = -4 \end{cases}$$

$$\begin{cases} c_1 + c_2 = 2\sqrt{2}e^{\pi/2} \\ -4 + e^{-\pi/2}\left(-\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right) = -4 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 2\sqrt{2}e^{\pi/2} \\ c_1 = c_2 \end{cases}$$

Continued

- So, $c_1 = c_2 = \sqrt{2}e^{\pi/2}$
- So, the solution is

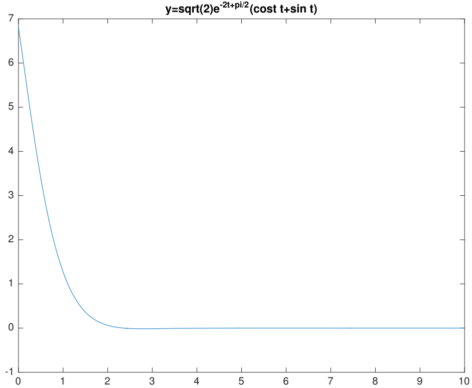
$$y = e^{-2t}\left(c_1 \cos t + c_2 \sin t\right)$$

$$= e^{-2t}\left(\sqrt{2}e^{\pi/2}\cos t + \sqrt{2}e^{\pi/2}\sin t\right)$$

$$= \sqrt{2}e^{-2t+\pi/2}\left(\cos t + \sin t\right)$$

- Repeat: the $y = y(t)$ has stabilized/dampened/ decaying oscillation.

Graph of $y = y(t)$: The exponential part $E(t) = e^{-2t+\pi/2}$ dampens (flattens) the graphs very quickly.



Example 3

Consider the IVP:

$$\begin{cases} y'' + 9y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

- Solve the problem
- Sketch the graph
- Describe the nature of the solution, as $t \to \infty$

Solution

- The CE: $r^2 + 9 = 0$
- Roots of the CE: $r_1 = 3i$, $r_2 = -3i$.
- By solution (8), the general solution

$$y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) = c_1 \cos 3t + c_2 \sin 3t$$

- Answer to the last part: the solution will be a STABLE oscillation; because there is no exponential part.

Continued

- Before we use the initial values, compute

$$y' = -3c_1 \sin 3t + 3c_2 \cos 3t$$

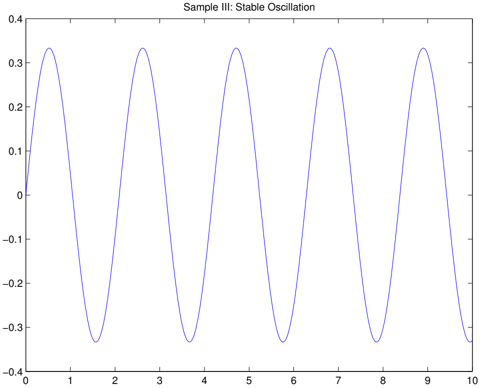
- Initial value conditions:

$$\begin{cases} y(0) = c_1 = 0 \\ y'(0) = 3c_2 = 1 \end{cases} \Longrightarrow \begin{cases} c_1 = 0 \\ c_2 = \frac{1}{3} \end{cases}$$

Continued

- So, the solution is
$$y = \frac{1}{3}\sin 3t$$
- Repeat: $y = y(t)$ has an STABLE oscillation.

Graph of $y = y(t)$:


Sample III: Stable Oscillation
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
0
1
2
3
4
5
6
7
8
9
10

On the Matlab Graph

- It took some trial and error to get a good graph.
- Following commands were used to get this graph:
  - t=[0:.01:10];
  - y=sin(3*t)/3;
  - plot(t,y), title('Sample III: Stable Oscillation')

Example 4

Consider the IVP:

$$\begin{cases} y'' + \pi^2 y = 0 \\ y(1) = 1 \\ y'(1) = 1 \end{cases}$$

- Solve the problem
- Sketch the graph
- Describe the nature of the solution, as $t \to \infty$

Solution

- The CE: $r^2 + \pi^2 = 0$
- Roots of the CE: $r_1 = \pi i$, $r_2 = -\pi i$.
- By solution (8), the general solution

$$y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) = c_1 \cos \pi t + c_2 \sin \pi t$$

- Answer to the last part: the solution will be a STABLE oscillation; because there is no exponential part.

Continued

- Before we use the initial values, compute

$$y' = -\pi c_1 \sin \pi t + \pi c_2 \cos \pi t$$

- Initial value conditions:

$$\begin{cases} y(1) = -c_1 = 1 \\ y'(1) = -\pi c_2 = 1 \end{cases} \implies \begin{cases} c_1 = -1 \\ c_2 = -\frac{1}{\pi} \end{cases}$$

Continued

- So, the solution is

$$y = c_1 \cos \pi t + c_2 \sin \pi t = -\cos \pi t - \frac{1}{\pi} \sin \pi t$$

- Repeat: $y = y(t)$ has an STABLE oscillation.

Graph of $y = y(t)$: