Chapter 2
First Order ODE
§2.2 Separable Equations

Satya Mandal

U. Kansas

Arrowtic *K*-Theory

Fall 2025

First Order ODE

- Recall the general form of the 1^{st} Order ODEs (FODE):

$$\frac{dy}{dt} = f(t, y) \tag{1}$$

where $f(t, y)$ is a function of both the independent variable t and the (unknown) dependent variable y.

- In section §2.1, we worked with Linear FODEs.
In this section we work with separable equations.

Separable Equations: Intro

- The equation (1) $\frac{dy}{dx} = f(x,y)$ can also be written as
$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$
with $M(x,y) := -f(x,y), N(x,y) = 1$
- There may be other choices of M, N. For example
$$\frac{dy}{dx} = \frac{x^2+y^2}{xy} \Longleftrightarrow -(x^2+y^2) + xy\frac{dy}{dx} = 0$$

Separable Equations: Definition

- A DE (1) $\frac{dy}{dx} = f(x,y)$ is said to be in separable form, if it can be written as $M(x) + N(y)\frac{dy}{dx} = 0$ or equivalently,

$$\text{in the differential form} \qquad M(x)dx + N(y)dy = 0 \qquad (2)$$

  where $M(x)$ is a function of x and $N(y)$ is a function of y.
- To solve them, we integrate

$$\int M(x)dx + \int N(y)dy = c \qquad c \text{ is arbitrary constant.} \qquad (3)$$

  We can use any anitiderivative $\int M(x)dx$ and $\int N(y)dy$.

Separable Equations: Initial value problem

- If initial value $y(x_0) = y_0$ is given then, we have choices

$$\int M(x)dx := \int_{x_0}^{x} M(s)ds \ \text{ and } \int N(x)dx := \int_{y_0}^{y} N(s)ds.$$

- With these choices, solution (3) reduces to

$$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = c$$

- Substituting $x = x_0, y = y_0$ we get $c = 0$.

Continued

- So, a form of the solution of the initial value problem:

$$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0 \qquad (4)$$

This form is particularly useful for numerical solutions.

Remarks

Here are some remarks:

- First, the equation 3 seems symmetric in x, y and does not seem to distinguish between independent and the dependent variable. The solution we find, by integrating, likely to be in the implicit form, which we solve to find y.
- A ODE (1) $\frac{dy}{dx} = f(x, y)$ sometimes could have a **constant solution** $y(x) = c$. This would be the case, if there is a constant y_0 such that $f(x, y_0) = 0$ for all x in the domain of $y = y(x)$. In this case, $y = y_0$ would be a constant solution.

Example I

Consider the initial value problem:

$$\begin{cases} y' = \frac{x}{y(x^2+1)} \\ y(0) = \sqrt{2} \end{cases}$$

- We have

$$\frac{dy}{dx} = \frac{x}{y(x^2+1)} \quad \Longrightarrow \quad ydy = \frac{x}{x^2+1}dx \quad \Longrightarrow$$

$$\int ydy = \int \frac{x}{x^2+1}dx + c \Longrightarrow \frac{y^2}{2} = \frac{1}{2}\ln\left|x^2+1\right| + c$$

Continued

- Substituting the initial values $y(0) = \sqrt{2}$, we have $c = 1$.
- So, the solution is given by the implicit formula

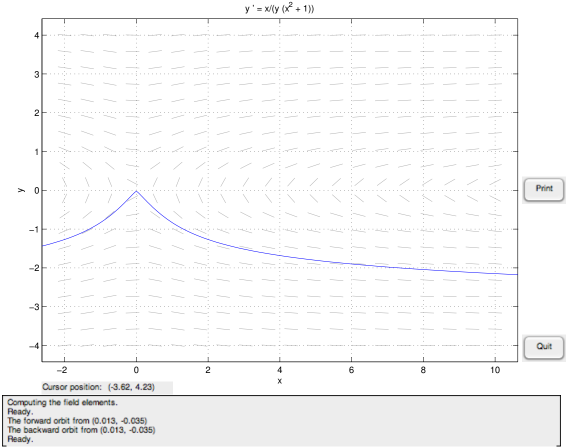
$$\frac{y^2}{2} = \frac{1}{2}\ln\left|x^2+1\right| + 1. \quad \text{So,} \quad y = \pm\sqrt{\ln\left|x^2+1\right| + 2}$$

- Since $y(0) = \sqrt{2}$, the finale solution is

$$y = \sqrt{\ln\left|x^2+1\right| + 2}$$

Solutions are valid everywhere.

The Direction fields and the integral curve:



Example II

Consider the initial value problem:

$$\begin{cases} y' = \frac{x(x^2+1)}{4y^3} \\ y(0) = \frac{1}{\sqrt{2}} \end{cases}$$

- We have

$$\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3} \Longrightarrow \int 4y^3 dy = \int x(x^2+1)dx + c \Longrightarrow$$

$$y^4 = \frac{x^4}{4} + \frac{x^2}{2} + c$$

Continued

- Use the initial value condition: $y(0) = \frac{1}{\sqrt{2}}$; we get

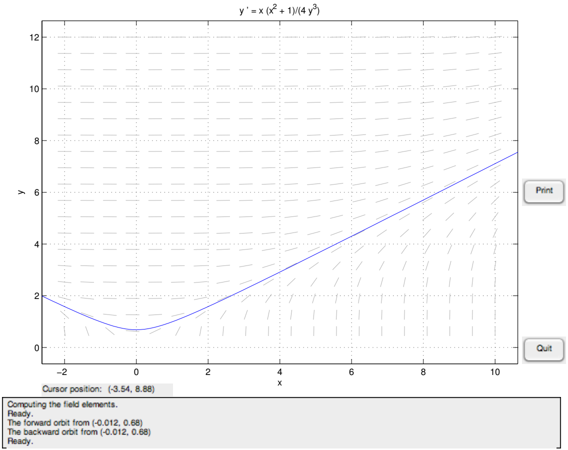
$$\frac{1}{4} = c \Longrightarrow y^4 = \frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{4} \Longrightarrow$$

$$y^4 = \frac{1}{4}(x^2+1)^2 \Longrightarrow y = \pm\sqrt{\frac{x^2+1}{2}}$$

- Since, $y(0) = \frac{1}{\sqrt{2}}$, the final answer:

$$(a) \quad y = \sqrt{\frac{x^2+1}{2}} \qquad\qquad (c) -\infty < x < \infty$$

The Direction fields and the integral curve:



Example III

- In last two problems we did not pay much attention to the range of x in which the solution is valid. A particular case is when $\frac{dy}{dx}$ is not defined for some values of x, y. Geometrically, this would mean, when the solution has a vertical tangent at these points.
- Additional work we do here is to specify intervals, where a solution is valid.

Consider the initial value problem:

$$\begin{cases} y' = \frac{3x^2}{3y^2-4} \\ y(1) = 0 \end{cases}$$

Continued

- We have
$$\frac{dy}{dx}=\frac{3x^2}{3y^2-4}\Longrightarrow\int(3y^2-4)dy=\int 3x^2dx+c\Longrightarrow$$
$$y^3-4y=x^3+c$$
- By the initial value condition $y(1)=0$ we have $0-0=1+c\Longrightarrow c=-1$. So, the final solution
$$y^3-4y=x^3-1 \qquad (5)$$
Further simplification does not seem worthwhile. So, solution $y=y(x)$ is given by the implicit equation $y^3-4y=x^3-1$

Continued

We further need to compute the interval, where the solution is valid.

- The equation $\frac{dy}{dx} = \frac{3x^2}{3y^2-4}$ is not defined, when $3y^2 - 4 = 0$ or $y = \pm\frac{2}{\sqrt{3}}$.
- So, the possible range of y are

$$\left(-\infty, -\frac{2}{\sqrt{3}}\right), \left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right), \left(\frac{2}{\sqrt{3}}, \infty\right)$$

- The initial y-value $y(1) = 0$. So, range of y is

$$\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \qquad \text{because } 0 \text{ is in it.}$$

Continued

- From, solution (5)

$$\begin{cases} y = \frac{2}{\sqrt{3}} \Longrightarrow x^3 - 1 = -\frac{16}{\sqrt{3}}, \\ y = -\frac{2}{\sqrt{3}} \Longrightarrow x^3 - 1 = \frac{16}{\sqrt{3}} \end{cases}$$

So, the domain of the solution is given by

$$-\frac{16}{\sqrt{3}} < x^3 - 1 < \frac{16}{\sqrt{3}} \qquad OR \qquad -2.02 < x < 2.17$$

The Direction fields and the integral curve: