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Complete Intersections and Cancellation of Projective Modules

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1. Introduction

An important open problem in the area of complete intersections is whether local complete intersection surfaces and higher dimensional varieties in affine $n-$spaces are set theoretic complete intersection. For the case of curves, this is a theorem due to Mohan Kumar [4]. In this paper we prove some results in this direction. One of the main results we prove here is the following:

Theorem 1.1. *Let k be an algebraically closed field and $R = k[X_1, \ldots, X_5]$ be polynomial ring and $\varphi : R \to A = A'[X]$ be a surjective map onto a polynomial algebra over a smooth algebra A' of dimension 2. Let I be a the kernel of φ. and $K = C_1(\omega_I)$. If $rK^2 = 0$ for some positive integer r then I is set theoretic complete intersection.*

The paper of Bloch-Murthy-Szpiro [1] contains some of the basic ingredients responsible for some of the recent developments in the complete intersection area. The above theorem extends the following theorem of Bloch-Murthy-Szpiro [1].

Theorem 1.2. *Let $R = k[X_1, X_2, X_3, X_4]$ be a polynomial ring over an algebraically closed field k and I be an ideal of height 2 in R and $A = R/I$, such that $X = Spec(R/I)$ is smooth over k. Let $\omega_I = \Lambda^2(I/I^2)^{-1}$ and $K_X = C_1(\omega_I)$ be the canonical divisor of X. If $rK_X^2 = 0$ in the Chow group $CH^2(X)$, for some integer $r > 0$, then X is set-theoretically complete intersection.*

For (an increasing sequence of) positive integers n, Mohan Kumar [5] constructed smooth affine algebras A with $\dim A = n$ and stably free projective $A-$modules of rank $n-2$ that are not free. He also posed the following question.

Question 1.1. **(LRC-problem)** *Let A be a (smooth) affine algebra of dimension n over an algebraically closed field k. Does the projective $A-$modules of rank $n-1$ have the cancellation property? That is, for a projective $A-$modul P with $rank(P) = n-1$, does $P \oplus A \approx Q \oplus A \Rightarrow P \approx Q$?*

We also investigate whether an affirmative answer to LRC-problem would imply that the local complete intersection surfaces and higher dimensional varieties in the affine $n-$spaces are set theoretic complete intersection. If LRC-problem has an affirmative answer for 3-folds, then we prove the following three dimensional analogue of the above theorem of Bloch-Murthy-Szpiro.

Theorem 1.3. *Suppose k is an algebraically closed field and $R = R'[X]$ is a polynomial ring over an affine algebra R' with $\dim R' = 4$. Let I be a locally complete intersection ideal of R with $height(I) = 2$ and $A = R/I$ is smooth over k. Assume that all rank 2 projective $A-$modules have cancellation property. Suppose $rK^2 = 0$ for some $r > 0$ where $K = C_1(\omega_I)$. Then I is set-theoretic complete intersection.*

We thank M. P. Murthy and N. Mohan Kumar for many useful discussions.

2. Main Results

Before we go into our main results we state the following monic polynomial version of a theorem of Ferrand and Szpiro.

Lemma 2.1. *Let $R = A[X]$ be a polynomial ring over a noetherian commutative ring A and J be a locally complete intersection ideal of height 2 in R. Assume that J contains a monic polynomial and there is a surjective map $J/J^2 \to \omega_J = Ext^2(R/J, R)$. Then J is set theoretically generated by 2 elements.*

The proof of this lemma can be found in [3]. We now prove a lemma which is a three dimensional analogue of a result in [1].

Lemma 2.2. *Let $R = R'[X]$ be polynomial ring over a noetherian commutative ring R' with $\dim R' = 4$ (so $\dim R = 5$). Let I be a locally complete intersection ideal of height 2 in R that contains a monic polynomial and $A = R/I$. Assume that all rank 2 projective $A = R/I$ modules have cancellation property. Then*

(1) $I/I^2 \approx A \oplus \omega^{-1}$

(2) Consider the following conditions

(a) ω is generated by two elements.

(b) $I/I^2 \approx \omega \oplus \omega^{-2}$

(c) ω^{-2} *is generated by 2 elements.*

Then a) $\Rightarrow$ *b)* $\Rightarrow$ *c).*

Proof. Here we have dim $A = 3$. We have an exact sequence

$$0 \to P \to R^n \to I \to 0$$

where P is a projective $R-$module of rank $n-1$. Since we can assume that $rank P = n-1 > \dim R$ and because I has a monic polynomial f it follows that $P_f \approx R_f^{n-1}$.

Hence, from theorem of Quillen and Suslin it follows $P \approx R^{n-1}$.

Tensoring the above sequence with A we get the following exact sequence:

$$0 \to L \to A^{n-1} \to A^n \to I/I^2 \to 0.$$

So, $[I/I^2] = [L] + [A]$ in $K_0(A)$. Since rank 2 projective dim $A-$modules have cancellation property, we have $I/I^2 \approx L \oplus A$. We also have $\Lambda^2 I/I^2 = L = \omega^{-1}$. So, $I/I^2 \approx A \oplus \omega^{-1}$.

Now we prove the second statement:

a) $\Rightarrow$ **b)** : By 1) we have $I/I^2 \approx \omega^{-1} \oplus A$ and also $A^2 \approx \omega \oplus \omega^{-1}$. So,

$$\begin{aligned} I/I^2 \oplus A &\approx (A \oplus \omega^{-1}) \oplus A \approx \omega^{-1} \oplus A^2 \approx \omega^{-1} \oplus (\omega \oplus \omega^{-1}) \\ &\approx \omega \oplus (\omega^{-1} \otimes A^2) \approx \omega \oplus \omega^{-1} \otimes (\omega \oplus \omega^{-1}) \approx \omega \oplus A \oplus \omega^{-2}. \end{aligned}$$

Since all rank 2 projective $A-$modules have cancellation property, we have $I/I^2 \approx \omega \oplus \omega^{-2}$.

b) $\Rightarrow$ **c)** : By 1) and b) it follows that

$$A \oplus \omega^{-1} \approx \omega \oplus \omega^{-2}.$$

By dualizing we get

$$A \oplus \omega \approx \omega^{-1} \oplus \omega^2.$$

Now adding A to both sides and using 1) we have:

$$A^2 \oplus \omega \approx (A \oplus \omega^{-1}) \oplus \omega^2 \approx I/I^2 \oplus \omega^2 \approx (\omega \oplus \omega^{-2}) \oplus \omega^2$$

Since we have assumed that rank 2 projective $A-$modules have cancellation property, by cancelling ω we get $A^2 \approx \omega^{-2} \oplus \omega^2$. So, ω^2 is generated by 2 elements.

Following theorem relates the cancellation property of rank 2 projective modules with set theoretic complete intersection property of height 2 locally complete intersection ideals. □

Theorem 2.3. *Let $R = R'[X]$ be a polynomial ring over a noetherian commutative ring R' with $\dim R' = 4$ (so $\dim R = 5$). Let $I \subseteq R$ be a locally complete intersection ideal of height 2 that contains a monic polynomial and $A = R/I$. Assume that projective A−modules of rank 2 have the cancellation property. Let $\omega = Ext^2(A, R)$. Assume that ω^r is generated by 2 elements, for some integer $r \geq 1$. Then I is a set-theoretic complete intersection ideal.*

Proof. By the above theorem we have, $I/I^2 \approx A \oplus \omega^{-1}$. So, we have an exact sequence

$$0 \to A \xrightarrow{f} I/I^2 \to \omega^{-1} \to 0,$$

where $f \in I$. So, $I = Rf + I'$, where $I^2 \subseteq I' \subseteq I$ is an ideal and $I'/I^2 \approx \omega^{-1}$. Write $J = I^r + Rf$. We will see that J is locally complete intersection ideal. For $\wp \in Spec(R)$ containing I, let $I_\wp = (f, g)$, where image of g generate ω^{-1}. So, $J_\wp = (f, g)^r + R_\wp f$. Hence $J_\wp = (g^r, f)$, is complete intersection.

Let L be the cokernel of the map

$$R/J \xrightarrow{f} J/J^2.$$

Since, locally $J_\wp$ is generated by (f, g^r), as in the above paragraph, we have L is locally generated by one element. So, L is a line bundle.

Since $\sqrt{I} = \sqrt{J}$, all rank 2 projective R/J−modules also have cancellation property. By the above lemma, we have $J/J^2 \approx R/J \oplus \omega_J^{-1}$ and $L \approx \omega_J^{-1}$. It follows that

(1) $J/J^2 \approx (R/J)f \oplus \omega_J^{-1}$.
(2) $\omega_J^{-1} \approx J/(J^2 + Rf)$.

Consider the exact sequence

$$0 \to R/J \xrightarrow{f} J/J^2 \to \omega_J^{-1} \to 0.$$

By tensoring the sequence with R/I, we get

$$0 \to R/I \xrightarrow{f} J/IJ \to \omega_J^{-1} \otimes R/I \to 0$$

So,

$$\omega_J^{-1} \otimes R/I \approx J/(IJ, f) \approx (I^r, f)/(I^{r+1}, f) \approx \omega_I^{-r},$$

which is 2 generated. So, ω_J^{-1} is also 2 generated. Hence $A^2 \approx \omega_J^{-1} \oplus \omega_J$. By lemma 2.2 $2a) \Rightarrow 2b)$, it follows that there is a surjective map

$$J/J^2 \to \omega_J.$$

Now it follows from lemma 2.1 that J is set theoretically generated by 2 elements. This completes the proof of this theorem.

Now we give a criterion for locally complete intersection ideal of height 2, in the polynomial ring R, to be ideal theoretically generated by $\dim R - 2$ elements. □

Theorem 2.4. *Let $R = R'[X]$ be the polynomial ring over a noetherian commutative R'. Assume $\dim R = n \geq 3$. Suppose I is a locally complete intersection ideal of height 2 that contains a monic polynomial and $A = R/I$. Assume that all projective A–modules of rank $n-3$. have cancellation property. Write $\omega = Ext^2(A, R)$. Then the following conditions are equivalent:*

(1) I is generated by $n-2$ elements,
(2) I/I^2 is generated by $n-2$ elements,
(3) ω is generated by $n-3$ elements.

Proof. **(1) ⇒ 2)** : This is obvious.

(2) ⇒ 3) : We have the following exact sequence:

$$0 \to P \to R^m \to I \to 0.$$

Since I has projective dimension 1, P is projective. We can also assume that $rank(P) > dim R$. Since I has a monic polynomial, we have $P_f \approx R_f^{m-1}$ for some monic polynomial $f \in I$. So, by the theorem of Quillen and Suslin, $P \approx R^{m-1}$.

By tensoring the above sequence with $A = R/I$, we get the following exact sequence:

$$0 \to L \to A^{m-1} \to A^m \to I/I^2 \to 0.$$

It follows that

$$L \approx \Lambda^2 I/I^2 \approx \omega^{-1}.$$

So, $[I/I^2] = [\omega^{-1} \oplus A]$ and hence $I/I^2 \oplus A^k \approx \omega^{-1} \oplus A^{k+1}$ or some k.

Since I/I^2 is generated by $n-2$ elements, we have a surjective map $A^{n-2+k} \to \omega^{-1} \oplus A^{k+1}$.

Therefore,

$$Q \oplus \omega^{-1} \oplus A^{k+1} \approx A^{n-2+k}.$$

Since $n-3$–projective modules have cancelltion property we get, $A^{n-3} \approx Q \oplus \omega^{-1}$. By dualizing this equality, we get ω is generated by $n-3$ elements.

(3) ⇒ 1) : Since ω is $n-3$ generated, by the argument of Serre (see [7]), there is an exact sequence

$$0 \to R^{n-3} \to P \to I \to 0.$$

Since I has a monic polynomial, we have P_f is free for some monic polynomial f. So, by the theorem of Quillen and Suslin, $P \approx R^{n-2}$. So, I is generated by $n-2$ elements. □

Theorem 2.5. *Let R' be a smooth affine algebra, with $\dim R' = n-1$, over an algebraically closed field k, with trivial differential module $\Omega_{R'/k} \approx R'^{n-1}$. Let $R = R'[X]$ be the polynomial ring. Assume $n = \dim R \geq 5$. Let I be an ideal of R so that $A = R/I$ is smooth and let I contain a monic polynomial. Assume that $n-2 \geq \dim A + 2$ or $height(I) = 1, 2$. In case $height(I) = 1$, assume that all projective A−modules of rank $n-2$ have cancellation property. Then the following conditions are equivalent:*

1) I is generated by $n-2$ elements.
2) I/I^2 is generated by $n-2$ elements.
3) $\Omega_{A/k}$ has a free direct summand of rank 2.

Proof. **1)** $\Rightarrow$ **2)** : Obvious.
2) $\Rightarrow$ **3)** : Consider the sequence

$$0 \to I/I^2 \to \Omega_{R/k}/I\Omega_{R/k} \to \Omega_{A/k} \to 0.$$

The sequence is exact. Also note that

$$\Omega_{R/k} \approx \Omega_{R'/k} \otimes R \oplus RdX \approx R^n.$$

So, it follows that $I/I^2 \oplus \Omega_{A/k} \approx A^n$.

By 2), we have $I/I^2 \oplus Q \approx A^{n-2}$. So, in $K_0(A)$, we have $[A^2] + [Q] = [\Omega_{A/k}]$. By cancellation theorem of Suslin, we have $\Omega_{A/k} \approx A^2 \oplus Q$.
3) $\Rightarrow$ **2)** : We have $\Omega_{A/k} = A^2 \oplus P$ and $I/I^2 \oplus \Omega_{A/k} \approx A^n$. So,

$$I/I^2 \oplus A^2 \oplus P \approx A^n.$$

By Suslin's cancellation theorem (or by hypothesis, when $height(I) = 1$) it follows that

$$I/I^2 \oplus P \approx A^{n-2}.$$

2) $\Rightarrow$ **1)** : If $n-2 \geq \dim(A) + 2$, then it follows from [2] that I is $n-2$ generated. If $height(I) = 1$ then I is one generated. If $height(I) = 2$ it follows from the above theorem. □

3. Smooth 3-folds in A^5

Recall that for any smooth affine variety X over an algebraically closed field k, the Chow group $A_0(X) = A^n(X)$ is torsion free (see [9] and [7]).

Notation 3.1. *Let A a smooth affine algebra over an algebraically closed field k and $X = Spec(A)$.*

(1) $A^r(X)$ will denote the Chow group of cycles of codimension r.
(2) For a projective $A-$module P, $C_i(P) \in A^i(X)$ will denote the $i-th$ Chern class of P.

Following is a restatement of the theorem of Murthy and Mohan Kumar [6] in our context of LRC-question.

Theorem 3.2. *Let $X = Spec(A)$ be a smooth affine 3-fold over an algebraically closed field k. Assume that projective $A-$modules of all rank have cancellation property. Then, for any two projective modules P and Q of same rank, $P \approx Q$ if and only if $C_i(P) = C_i(Q)$ for $i = 1, 2, 3$.*

Proof. ($\Rightarrow$) This implication is obvious.

($\Leftarrow$) The theorem of Mohan Kumar and Murthy [6] implies that P and Q are stably isomorphic. Now $P \approx Q$ by the cancellation property. □

Theorem 3.3. *Let $X = Spec(A)$ be a smooth affine 3-fold over an algebraically closed field k. Assume that projective $A-$modules of rank 2 have cancellation property. Let L be a projective $A-$module of rank one. Then the following conditions are equivalent:*

(1) L is generated by 2 elements.
(2) $C_1(L)^2 = 0$ in $A^2(X)$.
(3) $L \oplus L^{-1} \approx A^2$.

Proof.

1) $\Rightarrow$ 2) : Since L is two generated, we have $L \oplus L^{-1} \approx A^2$. So, $C(L \oplus L^1) = 1$. Hence $(1 + C_1(L))(1 - C_1(L)) = 1$ and $C_1(L)^2 = 0$.

2) $\Rightarrow$ 3)It follows that the total Chern class $C(L \oplus L^{-1}) = 1$. Hence $[L] \oplus [L^{-1}] = [A^2]$ in $K_0(A)$ and by cancellation $L \oplus L^{-1} \approx A^2$.

3) $\Rightarrow$ 1) : This implication is obvious. □

Theorem 3.4. *Suppose k is an algebraically closed field and $R = R'[X]$ a polynomial ring over an affine algebra R' with $\dim R' = 4$. Let I be a locally complete intersection ideal in R of height 2 that contains a monic polynomial. Assume $A = R/I$ is smooth and rank 2 projective $A-$modules have cancellation property. Then the following conditions are equivalent:*

(1) $K^2 = 0$ in $A^2(X)$ (where $K = C_1(\omega_I)$).
(2) ω_I is two generated.
(3) I is generated by 3 elements.

In all these cases, we have $I/I^2 \approx \omega_I \oplus \omega_I^{-2}$ *and that* I *is a set-theoretic complete intersection ideal.*

Proof. 1) $\Rightarrow$ 2) and 2) $\Rightarrow$ 1) follows from above theorem. 2) $\Rightarrow$ 3) and 3) $\Rightarrow$ 2) follows from theorem 2.4.

By lemma 2.2, we have $I/I^2 \approx A \oplus \omega^{-1}$. Since $K^2 = 0$, the total Chern class $C(I/I^2) = C(A \oplus \omega_I^{-1}) = C(\omega_I \oplus \omega_I^{-2})$. By cancellation $I/I^2 \approx \omega_I \oplus \omega_I^{-2}$. Also I is set-theoretic complete intersection by Theorem 2.3. So, the proof of this theorem is complete. □

Theorem 3.5. *Suppose* k *is an algebraically closed field and* $R = R'[X]$ *is a polynomial ring over an affine algebra* R' *with* $\dim R' = 4$. *Let* I *be a locally complete intersection ideal of* R *with* $height(I) = 2$ *and* $A = R/I$ *is smooth over* k. *Assume that all rank 2 projective* $A-$*modules have cancellation property. Suppose* $rK^2 = 0$ *for some* $r > 0$ *where* $K = C_1(\omega_I)$. *Then* I *is set-theoretic complete intersection.*

Proof. We can assume that $r^2K^2 = 0$. Let $L = \omega^r$. Then the total Chern classes

$$C(L \oplus L^{-1}) = (1 + rK)(1 - rK) = 1.$$

So, $L \oplus L^{-1} \approx A^2$. So, ω_I^r is two generated. By Theorem 2.3, I is a set-theoretic complete intersection. □

Following corollary follows from the above thoerem.

Corollary 3.6. *Let* k *be an algebraically closed field and* $R = k[X_1, \ldots, X_5]$ *be polynomial ring and* $\varphi : R \to A = A'[X]$ *be a surjective map onto a polynomial algebra over a smooth algebra* A' *of dimension 2. Let* I *be a the kernel of* φ. *and* $K = C_1(\omega_I)$. *If* $rK^2 = 0$ *for some positive integer* r *then* I *is set theoretic complete intersection.*

Proof. Note that rank 2 projective $A-$modules have cancellation property. Now the corollary follows from the above theorem. □

Theorem 3.7. *Let* $R = k[X_1, \ldots, X_5]$ *be a polynomial ring over an algebraically closed field* k *and* I *be a locally complete intersection ideal of height 2. Assume* $A = R/I$ *is smooth and rank 2 projective* $A-$*modules have cancellation property. If* $X = spec(A)$ *and* $rK_X^2 = 0$ *for some positive integer* r, *then* X *is set-theoretic complete intersection.*

Proof. Since $rK_X^2 = 0$ for some positive integer r, the theorem follows from the above theorem. □

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