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Vanishing of Euler class groups

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Abstract

In this paper, we prove some theorems about vanishing of Euler class groups. For example, suppose $B = A[X, f^{-1}]$ where $A[X]$ is a polynomial ring, $f \in A[X]$ is a non-zero divisor and $\dim(B) = \dim(A) + 1 \geqslant 3$. Then we prove that the Euler class group $E(B, L) = 0$ for any rank one projective B module L.

Keywords: Noetherian rings; Euler class groups

1. Introduction

In this paper we prove some results about vanishing of Euler class groups. For smooth affine algebras over fields, the Euler class group was introduced by M.V. Nori. Subsequently, for any noetherian commutative ring A and rank one projective modules L the Euler class group $E(A, L)$ were defined by Bhatwadekar and Raja Sridharan. We will use the definition of Bhatwadekar and Raja Sridharan [BRS1]. These groups were defined to study obstructions for projective modules P of top rank (i.e. $rank(P) = \dim A$) to have a rank one free direct summand.

In Section 3, we prove that for $B = A[X, 1/f]$, where $f \in A[X]$ is a non-zero divisor in a polynomial ring $A[X]$ over a noetherian commutative ring with $\dim(B) = \dim(A) + 1 = n$, and for a line bundle L over $Spec(B)$, the Euler class group $E(B, L) = 0$. When $L = B = A[X]$, the theorem is an easy consequence of the proof of the main theorem in [Ma] and needs a serious proof in this case of localization rings.

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In Section 4, we prove several theorems regarding equivalent conditions, for the Euler class group to vanish. For example, we prove that if all the cycles in the Cohen–Macaulay locus vanish, then the Euler class group vanishes. In Section 2, we will discuss some preliminaries.

For the definitions of Euler class groups and weak Euler class groups, the reader is referred to the paper of Bhatwadekar and Sridharan [BRS1] and for notations to [DM1,DM2]. For a noetherian commutative ring A with $\dim(A) = n \geqslant 2$ and a rank one projective modules L the Euler class group will be denoted by $E(A, L)$ and the weak Euler class group by $E_0(A, L)$.

2. Preliminaries

First, we will recall the following patching lemma of Quillen [Q].

Lemma 2.1. (Quillen [Q].) *Let A be a commutative ring and R be an A-algebra. Suppose $f \in A$ and θ be an unit in $1 + TR_f[T]$, where $R[T]$ is the polynomial ring over R in the variable T. Then there is an integer k, such that for $g_1, g_2 \in A$, whenever $g_1 - g_2 \in f^k A$, there is a unit ψ in $1 + fTR[T]$ such that $\psi_f(T) = \theta(g_1T)\theta(g_2T)^{-1}$.*

The following is a version of the patching lemma of Plumstead [P].

Lemma 2.2. (Plumstead [P].) *Let A be a commutative noetherian ring and M be an A-module. Suppose $As + At = 1$. Let $\alpha : M_{st} \to M_{st}$ be an isomorphism that is isotopic to the identity. Then we can find isomorphisms $\eta_1 : M_t \to M_t$ and $\eta_2 : M_s \to M_s$ such that*

- $\alpha = (\eta_2)_t(\eta_1)_s$,
- $\eta_1 \equiv \mathrm{Id}$ *(modulo s),*
- $\eta_2 \equiv \mathrm{Id}$ *(modulo t).*

Proof. Since α is isotopic to identity there is an isomorphism

$$\theta(T) : M_{st}[T] \xrightarrow{\sim} M_{st}[T]$$

such that $\theta(0) = \mathrm{Id}$ and $\theta(1) = \alpha$.

Write $R = End(M_t)$. So, R is an A-algebra. So, θ is an unit in $1 + TR_s[T]$. By 2.1, there is an integer $k \geqslant 0$ such that for $g_1, g_2 \in A$, whenever $g_1 - g_2 \in s^k A$, there is a unit ψ in $1 + sTEnd(M_t)[T]$ such that $\theta(g_1T)\theta(g_2T)^{-1} = \psi_s(T)$.

Again taking $R' = End(M_s)$, we consider θ as an element in $1 + TR'_t[T]$. So, there is an integer $k \geqslant 0$ such that for $g_1, g_2 \in A$, whenever $g_1 - g_2 \in t^k A$, there is a unit ψ in $1 + tTEnd(M_s)[T]$ such that $\theta(g_1T)\theta(g_2T)^{-1} = \psi_t(T)$.

Now we can take the same integer k for both the statements above. We can write $1 = \lambda s^k + \mu t^k$. Now

$$\theta(T) = \left[\theta(T)\theta\left(\lambda s^k T\right)^{-1}\right]\left[\theta\left(\lambda s^k T\right)\theta(0)^{-1}\right].$$

Since $1 - \lambda s^k \in t^k A$, there is an unit ψ_2 in $1 + tTEnd(M_s)[T]$ such that $(\psi_2)_t(T) = \theta(T)\theta(\lambda s^k T)^{-1}$.

Similarly, since $\lambda s^k - 0 \in s^k A$ there is an unit $\psi_1 \in 1 + sTEnd(M_t)[T]$ such that $(\psi_1)_s(T) = \theta(\lambda s^k T)^{-1}\theta(0)^{-1}$. Therefore,

$$\theta(T) = (\psi_2)_t(T)(\psi_1)_s(T).$$

Substituting $T = 1$ we have $\alpha = (\eta_2)_t(\eta_1)_s$, where $\eta_i = \psi_i(1)$, for $i = 1, 2$. This completes the proof of this lemma. $\square$

The referee suggested the following version of Quillen's argument [Q] regarding extendibility of modules.

Proposition 2.3. *Let A be a noetherian commutative ring and $R = A[X]$ be the polynomial ring. Suppose N is a finitely generated R-module and*

$$Q = \{s \in A\colon\ N_s \approx E \otimes R_s \text{ where } E \text{ is } A_s\text{-projective}\}.$$

Then Q is an ideal.

Proof. Clearly, $0 \in Q$ and $as \in Q$ for all $a \in A$ and $s \in Q$. So, we need to prove that $s, t \in Q \Rightarrow s + t \in Q$. Assume $s, t \in Q$. We will prove $s + t \in Q$. By replacing A by A_{s+t} we may assume $s + t = 1$. Write $M_1 = E_1 \otimes R_s$ and $M_2 = E_2 \otimes R_t$ where E_1 is a projective R_s-module and E_2 is a projective R_t-module. Let

$$f_1 : M_1 \xrightarrow{\sim} N_s \quad \text{and} \quad f_2 : M_2 \xrightarrow{\sim} N_t$$

be two isomorphisms and

$$\Theta = \left(f_2^{-1}\right)_s (f_1)_t : (M_1)_t \xrightarrow{\sim} (M_2)_s.$$

Let "overline" denote "modulo X." Then $\overline{\Theta} : (E_1)_t \xrightarrow{\sim} (E_2)_s$ is an isomorphism. Consider the following two fiber product diagrams:

$$\begin{tikzcd} E_0 \arrow[rr, "q_2"] \arrow[d, "q_1"] & & E_2 \arrow[d] \\ E_1 \arrow[r] & (E_1)_t \arrow[r, "\overline{\Theta}"] & (E_2)_s, \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, "p_2"] \arrow[d, "p_1"] & & M_2 \arrow[d] \\ M_1 \arrow[r] & (M_1)_t \arrow[r, "\Theta"] & (M_2)_s. \end{tikzcd}$$

Clearly, E_0 is a projective A-module and E is a projective R-module. Now use standard arguments to show $E_0 \otimes R \approx E \approx N$ (see, for example, [Ma1]). This completes the proof. $\square$

We quote the following version of Swan's Bertini theorem from [BRS2, p. 291].

Theorem 2.4. *Let A be a geometrically reduced affine ring over an infinite field and P be a projective A-module of rank r. Let $(\alpha, s) \in P^* \oplus A$. Then there is an element $\beta \in P^*$ such that if $I = (\alpha + s\beta)(P)$, then*:

(1) *Either $I_s = A_s$ or I_s is an ideal of height r such that $(A/I)_s$ is a geometrically reduced ring.*
(2) *If $r < \dim A$ and A_s is geometrically integral, then $(A/I)_s$ is a geometrically integral.*
(3) *If A_s is smooth, then $(A/I)_s$ is smooth.*
(4) *In particular, if $J = (a_1, \dots, a_r, s)$ is an ideal of A then there exist $d_1, \dots, d_r \in A$ such that if $J = (a_1 + sd_1, \dots, a_r + sd_r)$, then I_s satisfies properties* (1)–(3).

The following is the theorem of Grothendieck [EGA, p. 158] on the openness of the Cohen–Macaulay locus.

Theorem 2.5. (Grothendieck [EGA].) *Let A be noetherian commutative ring and M be a finitely generated A-module. Assume that A is image of a regular ring. Define the map*

$$coDepth : Spec(A) \to \mathbb{Z} \quad \text{given by } coDepth(\wp) = \dim(M_\wp) - depth(M_\wp).$$

Then coDepth is upper semi continuous. That means, for any $\wp_0 \in Spec(A)$ there is an open neighborhood U of $\wp_0$ such that

$$\wp \in U \quad \Rightarrow \quad coDepth(\wp) \leqslant coDepth(\wp_0).$$

In fact,

$$\mathcal{U} = \{\wp \in Spec(A):\ coDepth(\wp) \leqslant n\}$$

is open for all $n \in \mathbb{Z}$.

In particular, the Cohen–Macaulay locus $CM(A)$ is non-empty open in $Spec(A)$.

Proof. For the benefit of the reader, we will sketch the proof. Let B be a regular ring and $\varphi : B \to A$ be a surjective homomorphism and $\Phi : Spec(A) \to Spec(B)$ be the induced map. Consider M as a B-module.

It is easy to see that the map, $coDepth : Spec(A) \to \mathbb{Z}$ extends to $Spec(B)$. So, replacing A by B, we can assume that A is a regular ring.

Since A is regular, we can use Auslander–Buchsbaum formula. It follows that, for $\wp \in Spec(A)$ we have

$$coDepth(\wp) = \dim(M_\wp) - depth(M_\wp) = projDim(M_\wp) - [height(I_\wp)]$$

where $I = ann(M)$.

It is easy to see that, for any integer $n \in \mathbb{Z}$ the sets

$$U(n) = \{\wp \in Spec(A):\ projDim(M_\wp) \leqslant n\}$$

and

$$V(n) = \{\wp \in Spec(A):\ height(I_\wp) \geqslant n\}$$

are open. This completes the proof. □

3. Main results

The following is the main theorem on vanishing of Euler class group of polynomial rings.

Theorem 3.1. *Let $R = A[X]$ be a polynomial ring over a commutative noetherian ring A and $B = A[X, 1/f]$, where $f \in R$ is a non-zero divisor. Assume $\dim B = \dim A + 1 \geqslant 3$. Let $\mathcal{L}$ be rank one projective B-module. Then $E(B, \mathcal{L}) = 0$ and $E_0(B, \mathcal{L}) = 0$.*

Proof. Since there is a surjective map $E(B, \mathcal{L}) \to E_0(B, \mathcal{L})$, we will only prove $E(B, \mathcal{L}) = 0$. We also assume that A is reduced [BRS1, Corollary 4.6] and A has no non-trivial idempotent.

We will consider $\mathcal{L}$ as an invertible ideal and let $L = R \cap \mathcal{L}$. Write $n = \dim B$. We will write $\mathcal{F} = B^{n-1} \oplus \mathcal{L}$ and $F = R^{n-1} \oplus L$. Let $\mathcal{I}$ be a primary ideal of B with $height(\mathcal{I}) = n$ and let $\omega : \mathcal{F}/\mathcal{I}\mathcal{F} \to \mathcal{I}/\mathcal{I}^2$ be a local $\mathcal{L}$-orientation of $\mathcal{I}$. We will prove that $(\mathcal{I}, \omega) = 0$ in $E(B, \mathcal{I})$.

Let

$$Q = \{s \in A:\ L_s \approx E \otimes R_s \text{ where } E \text{ is } A_s\text{-projective}\}.$$

By Proposition 2.3, Q is an ideal of A. Since L is an ideal and A is reduced, we have $height(Q) \geqslant 1$.

Let $I = \mathcal{I} \cap R$. Note that I is a primary ideal of R with $height(I) = n$. So, $m_0 = \sqrt{I}$ is a maximal ideal of height n. Write

$$\mathcal{P} = \left\{\wp \in Spec(R):\ I \not\subseteq \wp \text{ and } height(\wp) < n \text{ or } Q \subseteq \wp\right\}.$$

So,

$$\mathcal{P} = \left\{\wp \in Spec(R):\ m_0 \neq \wp \text{ and } height(\wp) < n \text{ or } Q \subseteq \wp\right\}.$$

Note that there is a generalized dimension function $d : \mathcal{P} \to \{0, 1, 2, \ldots\}$ such that $d(\wp) \leqslant n - 1$ for all $\wp \in \mathcal{P}$.

Let $\gamma : \mathcal{F} \to \mathcal{I}$ be any lift of ω.

Let $\beta_0 : F \to I$ be such that $\gamma = \beta_0/f^{2k}$. Since $(\mathcal{I}, \omega) = f^{2k}(\mathcal{I}, \omega)$ [BRS1, Lemma 5.4], replacing ω by $f^{2k}\omega$, we can assume that $\gamma = \beta_0/1$.

Note that $Hom(F, R)_\wp = I^2 Hom(F, R)_\wp$ for all $\wp \in \mathcal{P}$.

So, $\beta = \beta_0 + \beta_1$ is basic in $Hom(F, R)$ on $\mathcal{P}$ for some $\beta_1 \in I^2 Hom(F, R)$. Consider the following commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & F/IF & \xrightarrow{\sim} & \mathcal{F}/\mathcal{I}\mathcal{F} \\ {\scriptstyle\beta}\downarrow & & \downarrow & & \downarrow{\scriptstyle\omega} \\ I & \longrightarrow & I/I^2 & \xrightarrow{\sim} & \mathcal{I}/\mathcal{I}^2. \end{array}$$

Therefore, β_f is a lift of ω and also $I = image(\beta) + I^2$. So, $image(\beta) = I \cap K$ for some ideal K of R with $I + K = R$, $height(K_f) \geqslant n$ and $QR_f + K_f = R_f$.

Let ω' be the local $\mathcal{L}$-orientation on K_f induced by β_f. Therefore,

$$(\mathcal{I},\omega)+(K_f,\omega')=0.$$

So, we will prove $(K_f,\omega')=0$. Let

$$K=K_1\cap K_2\cap\cdots\cap K_r\cap K_{r+1}\cap\cdots\cap K_t$$

be a irredundant primary decomposition of K, where $f\notin\sqrt{K_i}$ for $i=1,\dots,r$ and $f\in\sqrt{K_i}$ for $i=r+1,\dots,t$. Also note $height(K_i)=n$ for $i=1,\dots,r$. We have

$$(K_f,\omega')=\sum_{i=1}^{r}\big((K_i)_f,\omega_i\big)$$

where ω_i is induced by ω'. So, we will prove $((K_i)_f,\omega_i)=0$ for $i=1,\dots,r$.

Replacing (I,ω) by $((K_i)_f,\omega_i)$, we can additionally assume that $QR_f+I_f=R_f$. Since $I+Rf=R$, in fact, we have

$$QR+I=R.$$

The diagram above remains valid. By abuse of notation, the map $F/IF\to I/I^2$ will also be denoted by ω.

Write $J=I\cap A$. Since I has a monic polynomial, we have $Q+J=A$. Let 'overline' denote modulo I. Let $e_1,\dots,e_{n-1}$ be the standard basis of $R^{n-1}\subseteq F$. Let $e_n=(0,\dots,0,l)\in F$ be such that $L/IL=(A/I)l=\mathcal{L}/\mathcal{I}\mathcal{L}$.

Let $f_1\in I$ be such that $\overline{f_1}=\omega(\overline{e_1})$. We can assume f_1 is a monic polynomial. We will pick $f_2\in I$ such that $\overline{f_2}=\omega(\overline{e_2})$ and for any maximal ideal m, if $(J,f_1,f_2)\subseteq m$ then $I\subseteq m$. To do this, let $g_2\in I$ be such that $\overline{g_2}=\omega(\overline{e_2})$. Let $m_1,\dots,m_r,m_{r+1},\dots,m_l$ be the maximal ideals over (J,f_1) such that $I\nsubseteq m_i$ (that means, $m_i\neq\sqrt{I}$). Assume $g_2\notin m_i$ for $i=1,\dots,r$ and $g_2\in m_i$ for $i=r+1,\dots,l$. Pick $\lambda\in I^2\cap\bigcap_{i=1}^r m_i\setminus\bigcup_{i=r+1}^l m_i$. Write $f_2=g_2+\lambda$. Now, f_2 will satisfy this property.

For $i=3,\dots,n-1$ let $f_i\in I$ be any lift of $\omega(\overline{e_i})$. Let $\gamma:L\to I$ be any lift of $\omega|_{L/IL}$. (*Note, γ exists. A choice of γ could be the restriction $\beta|_L$ and for $i=3,\dots,n-1$ we could take $f_i=\beta(e_i)$.*)

Let $\varphi_0:F\to I$ be given by $f_1,\dots,f_{n-1},\gamma$. We claim that $\varphi_0(F)_{1+J}=I_{1+J}$.

Let m be a maximal ideal of R_{1+J} such that $\varphi_0(F)_{1+J}\subseteq m$. Since $f_1\in m$ is monic, we have $m\cap A_{1+J}$ is maximal and hence $J\subseteq m$. Therefore, $(J,f_1,f_2)\subseteq m$. Therefore, by choice, we have $I_{1+J}\subseteq m$. Since $I=\varphi_0(F)+I^2$, we have $I_{1+J}=\varphi_0(F)_{1+J}$.

So, we can pick an $s\in J$, such that the map $\varphi_1:F_{1+s}\to I_{1+s}$ given by $f_1,\dots,f_{n-1},\gamma$ is surjective. Since, $Q+J=A$, we can assume that $1+s\in Q\cap(1+J)$.

Since $Q+J=A$, we have L_{1+J} is extended from A_{1+J} and is an invertible (projective) ideal. Since A_{1+J} is semi-local, L_{1+J} is, in fact, free. Therefore, by modifying s, we can assume that L_{1+s} is free.

Let $\varphi_2:F_s\to I_s$ be a surjective map given by $(1,0,\dots,0)$.

Since $F_{s(1+s)}$ is free and f_1 is monic, by a theorem of Ravi Rao [R2], there is elementary matrix $\alpha\in Aut(F_{s(1+s)})$ such that $(\varphi_2)_{1+s}\alpha=(\varphi_1)_s$.

Consider the following fiber product diagram:

$$\begin{tikzcd}
P \arrow[rrr, "\pi_2"] \arrow[dr, "\varphi"] \arrow[dd, "\pi_1"'] & & & F_s \arrow[dr, "\varphi_2"] \arrow[dd] & \\
& I \arrow[rrr] \arrow[dd] & & & I_s \arrow[dd] \\
F_{1+s} \arrow[rr] \arrow[dr, "\varphi_1"] & & F_{s(1+s)} \arrow[r, "\alpha"] \arrow[dr, "\varphi_1"] & F_{s(1+s)} \arrow[dr, "\varphi_2"] & \\
& I_{1+s} \arrow[rr] & & I_{s(1+s)} \arrow[r, "\mathrm{Id}"] & I_{s(1+s)},
\end{tikzcd}$$

where P is the R-module obtained by patching F_{1+s} and F_s via α. The map φ obtained by the properties of fiber product diagrams. Note that φ is surjective.

We will construct an isomorphism $\Psi : F \to P$ such that $(\varphi\Psi)_{1+s} \equiv \varphi_1$ (modulo s). This will mean that $(\varphi\Psi)_f$ is a surjective lift of ω.

Since α is elementary, α is isotopic to the identity. Therefore, by 2.2, there are isomorphisms $\eta_1 : F_{1+s} \to F_{1+s}$ and $\eta_2 : F_s \to F_s$ such that

- $\alpha = (\eta_2)_{1+s}(\eta_1)_s$,
- $\eta_1 \equiv \mathrm{Id}$ (modulo s).

Now let $\psi_1 = (\pi_1)^{-1}_{1+s}\eta_1^{-1} : F_{1+s} \to P_{1+s}$ and $\psi_2 = (\pi_2)^{-1}_s\eta_2 : F_s \to P_s$. Then $(\psi_1)_s = (\psi_2)_{1+s}$. So, there is an isomorphism $\Psi : F \to P$ such that $\Psi_s = \psi_2$, and $\Psi_{1+s} = \psi_1$.

Now we claim $\varphi\Psi : F \to I$ is a surjective lift of ω. We will check that $\varphi_{1+s}\Psi_{1+s} : F_{1+s} \to I_{1+s}$ is a lift of ω. We have $\varphi_{1+s}\Psi_{1+s} = \varphi_{1+s}\psi_1 = \varphi_{1+s}(\pi_1)^{-1}_{1+s}\eta_1^{-1} = \varphi_1\eta_1^{-1}$. Since, $\eta_1^{-1} \equiv \mathrm{Id}$ (modulo s) and φ_1 is a lift of ω, the claim is established.

By localizing, we have

$$\phi = (\varphi\Psi)_f : \mathcal{F} \twoheadrightarrow \mathcal{I}$$

is a surjective lift of ω. Therefore, we have $(\mathcal{I}, \omega) = 0$ in $E(B, \mathcal{L})$ and the proof is complete. □

The following corollary is an immediate consequence of the above Theorem 3.1 and theorem of Bhatwadekar and Sridharan [BRS1, p. 199].

Corollary 3.2. *Let $R = A[X]$ be a polynomial ring over a commutative noetherian ring A and $B = A[X, 1/f]$, where $f \in R$ is a non-zero divisor. Assume $\dim B = \dim A + 1 = n \geqslant 3$. Let L be a rank one projective B-module. Suppose I is an ideal in B with height$(I) = n$ and $\omega : L \oplus B^{n-1} \twoheadrightarrow I/I^2$ is a surjective map. Then ω lifts to a surjection $\varphi : L \oplus B^{n-1} \twoheadrightarrow I$.*

Now assume that $1/(n-1)! \in B$. Let P be a projective B-module with rank$(P) = n$ and $\det(P) = L$. Let $\chi : L \xrightarrow{\sim} \bigwedge^n P$ be an isomorphism. Let I be an ideal in B with height$(I) = n$ and $\omega : P/IP \twoheadrightarrow I/I^2$ be a surjective homomorphism. Then there is a surjective homomorphism $\varphi : P \twoheadrightarrow I$ such that (I, ω) is obtained from (φ, χ) (see [BRS1, Corollary 4.3] for clarification).

Proof. Since $E(B,L)=0$, we have $(I,\omega)=0$. By [BRS1, Theorem 4.2], ω lifts to a surjection $\varphi: L\oplus B^{n-1}\twoheadrightarrow I$. Note that [BRS1, Theorem 4.2] does not require that $1/(n-1)!\in B$. This completes the proof. $\square$

For the later part, since $1/(n-1)!\in B$, Euler class $e(P,\chi)\in E(B,L)$ is defined. As $E(B,L)=0$, we have $e(P,\chi)=(I,\omega)=0$. Therefore the assertion follows from [BRS1, Corollary 4.3].

Remark 3.3. Let $R=A[X]$ be a polynomial ring over a commutative noetherian ring A and $B=A[X,1/f]$, where $f\in R$ is a non-zero divisor. Assume $\dim B=\dim A+1=n\geqslant 3$. It is a theorem of Ravi Rao [R1] that any projective B-module P with $rank(P)=n$, has a free direct summand. If $1/(n-1)!\in B$, then Euler classes of P are defined and hence the theorem of Rao [R1] follows from Theorem 3.1.

4. Equivalence theorems

In this section we prove some results regarding vanishing of Euler class groups with respect to vanishing of certain types of cycles.

Theorem 4.1. *Let A be a geometrically reduced affine algebra over an infinite field k with $\dim A=n$ and L be a line bundle over $Spec(A)$. Then the following are equivalent*:

(1) $E_0(A,L)=0$.
(2) *The cycle $(I)=0$ in $E_0(A,L)$ for all local complete intersection ideals I with $height(I)=n$.*
(3) *The cycle $(\mathfrak{m})=0$ in $E_0(A,L)$, for all smooth maximal ideals $m\in Spec(A)$ of height n.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious. So, we prove (3) $\Rightarrow$ (1). Let I be an ideal of height n such that I/I^2 is generated by n elements. We will prove that the cycle $(I)=0$ in $E_0(A,L)$. Write $F=L\oplus A^{n-1}$. Take any surjection from $\omega: F/IF\to I/I^2$ and lift it to $\alpha: F\to I$. Notice that α is not necessarily a surjection. However, I is generated by $image(\alpha)$ and some $s\in I^2$. By Swan's Bertini Theorem 2.4, there we can find $\alpha'\in Hom(F,A)$ such that, with $\beta=\alpha+s\alpha'$, we have $\beta(F)=I\cap J$ for some reduced ideal J of height n with $I+J=A$.

Therefore $J=\mathfrak{m}_1\cap\cdots\cap\mathfrak{m}_k$, where $\mathfrak{m}_i$ are maximal ideals. This results in $(I)+\sum_{i=1}^{k}(\mathfrak{m}_i)=(I\cap J)=0$ in $E_0(A,L)$. Now by hypothesis, $(\mathfrak{m}_i)=0$ for all i, yielding $(I)=0$. This completes the proof. $\square$

The following is the Euler class group version of the above theorem.

Theorem 4.2. *Let A be a geometrically reduced affine algebra over an infinite field k, with $\dim A=n\geqslant 2$ and L be a rank one projective module. We write $F=L\oplus A^{n-1}$. Then the following are equivalent*:

(1) $E(A,L)=0$.
(2) *The cycle $(I,\omega)=0$ in $E(A,L)$ for all local complete intersection ideals I with $height(I)=n$, and local orientation $\omega: F/IF\to I/I^2$.*
(3) *The cycle $(\mathfrak{m},\omega)=0$ in $E(A,L)$ for all smooth maximal ideal $\mathfrak{m}\in Spec(A)$, of height n, and local orientation $\omega: F/\mathfrak{m}F\to\mathfrak{m}/\mathfrak{m}^2$.*

Proof. We will only prove (3) $\Rightarrow$ (1). Let I be an of height n and $\omega : F/IF \to I/I^2$ be a surjective map (local orientation). Now let $\alpha : F \to I$ be a lift of ω. Then $I = (\alpha(F), s)$ for some $s \in I^2$. By Swan's Bertini Theorem 2.4 there is $\alpha' \in F^*$ such that if $\beta = \alpha + s\alpha'$ and $J = im(\beta)$ then:

(1) There is a reduced ideal J' of height n so that $I + J' = A$ and $J = I \cap J'$.
(2) Since J' is reduced, $J' = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$, where $\mathfrak{m}_i$ are maximal ideals with $A_{\mathfrak{m}_i}$ regular with dimension n.
(3) Notice that β is a lift of ω.
(4) We also have $(J, \omega_0) = 0$ in $E(A, L)$, where ω_0 is induced by β.

Therefore, in $E(A, L)$, we have $(I, \omega) + \sum_{i=1}^{k} (\mathfrak{m}_i, \omega_{\mathfrak{m}_i}) = (J, \omega_0) = 0$ where $\omega_{\mathfrak{m}_i} : F/\mathfrak{m}_i F \to \mathfrak{m}_i/\mathfrak{m}_i^2$ is the local orientation on $\mathfrak{m}_i$ induced by β. By hypothesis, each $(\mathfrak{m}_i, \omega_{\mathfrak{m}_i}) = 0$, leaving us with $(I, \omega) = 0$. This completes the proof. $\square$

The following are versions of the above theorems for rings that are images of regular rings.

Theorem 4.3. *Let A be an noetherian ring with* $\dim A = n$. *Assume that A is image of a regular ring. Let L be a projective A-module of rank one. Then the following are equivalent*:

(1) *For all local complete intersection ideals N where N is primary with height$(N) = n$ and N/N^2 is generated by n elements, $(N) = 0$ in $E_0(A, L)$.*
(2) *For all local complete intersection ideals J with height$(J) = n$ and J/J^2 is generated by n elements, $(J) = 0$ in $E_0(A, L)$.*
(3) *$E_0(A, L) = 0$.*

Proof. (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (3) Let $CM(A)$ denote the Cohen–Macaulay locus of $Spec(A)$. By Theorem 2.5, $CM(A)$ is non-empty and open. So, $Spec\, A \setminus CM(A) = V(I)$ for some ideal I. Since rings of dimension zero are Cohen–Macaulay, height of I is at least one.

Now suppose N is a primary ideal with $height(N) = n$, and N/N^2 is generated by n elements. We will prove that $(N) = 0$ in $E_0(A, L)$.

Write $F = L \oplus A^{n-1}$. There is a homomorphism $\varphi_0 : F \to N$, such that the induced map $F/NF \to N/N^2$ is surjective. So, $N = \varphi_0(F) + N^2$. Hence, $N = (\varphi_0(F), s)$ for some $s \in N^2$. Write

$$\mathcal{P} = \{\wp \in Spec(A) \colon height(\wp) < n \text{ or } (I \subseteq \wp \text{ and } N \nsubseteq \wp\}.$$

Note that $(\varphi_0, s) \in Hom(F, A) \oplus A$ basic on $\mathcal{P}$. Also there is a generalized dimension function $d : \mathcal{P} \to \mathbb{Z}$

$$d(\wp) < n = rank(F) \quad \text{for all } \wp \in \mathcal{P}.$$

Therefore, by theorem of Eisenbud and Evans, $\varphi = \varphi_0 + s\beta$ is basic on $\mathcal{P}$ for some $\beta \in Hom(F, A)$.

Write $J_0 = \varphi(F)$. Then $height(J_0) = n$ and $J_0 = N \cap J$ for some ideal J with $height(J) = n$ and $N + J = A$.

Suppose $\wp \in Spec(A)$ and $J \subset \wp$. Then $J_0 \subseteq \wp$ and $N \nsubseteq \wp$. Therefore $I \nsubseteq \wp$. Therefore $A_\wp$ is Cohen–Macaulay. This also implies that J is locally complete intersection ideal of height n.

Looking at a primary decomposition, $J = \bigcap_{i=1}^k N_i$ with N_i primary local complete intersection for all i. Now we have $N \cap (\bigcap_{i=1}^k N_i) = J_0$. Since $J_0 = \varphi(F)$, we have $(N) + \sum_{i=1}^k (N_i) = (J_0) = 0$ in $E_0(A, L)$. By (1), $(N_i) = 0$ for all i. Thus $(N) = 0$. Therefore $E_0(A, L) = 0$. So, the proof of the theorem is complete. □

Theorem 4.4. *Let A be an noetherian ring with* $\dim A = n$. *Assume that A is image of a regular ring. Let L be a projective A-module of rank one and $F = L \oplus A^{n-1}$. Then the following are equivalent*:

(1) *For all local orientations $\omega : F/NF \to N/N^2$ where N is primary local complete intersection ideal with height$(N) = n$, we have $(N, \omega) = 0$ in $E(A, L)$.*
(2) *For local orientations $\omega : F/JF \to J/J^2$ where J is local complete intersection ideal with height$(J) = n$, we have $(J, \omega) = 0$ in $E(A, L)$.*
(3) *$E(A, L) = 0$.*

Proof. The proof is similar to the proof of the above Theorem 4.3. The proofs of (3) ⇒ (2) and (2) ⇒ (1) are obvious.

(1) ⇒ (3) As before, the Cohen–Macaulay locus $CM(A)$ of $Spec(A)$ is open and $Spec\, A \setminus CM(A) = V(I)$ for some ideal I with $height(I) \geqslant 1$.

Now suppose N is a primary ideal with $height(N) = n$, and $\omega : F/NF \to N/N^2$ be a local orientations. We will prove that $(N, \omega) = 0$ in $E(A, L)$.

Let $\varphi_0 : F \to N$ be a lift of ω (that is not necessarily surjective). As in the proof of Theorem 4.3, we can find $s \in N^2$, and $\varphi = \varphi_0 + s\beta$ for some $\beta \in Hom(F, A)$ such that if $J_0 = \varphi(F)$ then $J_0 = N \cap J$ where J is local complete intersection ideal of height n and $J + N = A$.

Looking at the primary decomposition

$$J = \bigcap_{i=1}^{k} N_i$$

of J, where N_i is primary local complete intersection for all $i = 1, \ldots, k$. For $i = 1, \ldots, k$ let $\omega_i = \varphi \otimes A/N_i$ and let $\omega_0 = \varphi \otimes A/J_0$.

Note $(J_0, \omega_0) = 0$ in $E(A, L)$ and $\varphi \otimes A/N = \omega$. Now we have

$$N \cap \left(\bigcap_{i=1}^{k} N_i \right) = J_0.$$

Therefore

$$(N, \omega) + \sum_{i=1}^{k} (N_i, \omega_i) = (J_0, \omega_0) = 0.$$

By (1), $(N_i, \omega_i) = 0$ for all $i = 1, \ldots, k$. Thus $(N, \omega) = 0$. Therefore $E(A, L) = 0$. This completes the proof of the theorem. □

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