Chapter 5
System of 1^{st}-Order Linear ODE
§5.7 Repeated Eigenvalues

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Repeated Eigenvalues

- We continue to consider homogeneous linear systems with constant coefficients:

$$y' = Ay \quad \text{A is an n} \times \text{n matrix with constant entries} \tag{1}$$

- Now, we consider the case, when some of the eigenvalues (real or complex) are repeated.
- For an eigen value r of A, let

$$E(r) = NullSpace(A - rI) \quad \text{denote the eigen space of r}$$

Two Cases of higher multiplicity

Consider the system (1). Let r be an eigenvalue of A with multiplicity $m \geq 2$. (*Here r is real or complex.*) Two cases.

- **Case** $m = \dim E(r)$: there is a basis of m linearly independent eigenvectors:

$$\xi^{(1)}, \ldots, \xi^{(m)} \quad \text{of A.} \quad i.e. \quad (A - rI)\xi^{(i)} = 0.$$

- **Case** $k = \dim E(r) \leq m - 1$: There is a basis of n linearly independent eigenvectors (fewer than m)

$$\xi^{(1)}, \ldots, \xi^{(k)} \quad \text{of A} \quad k \leq m - 1$$

- If r is real, then the eigenvectors $\xi^{(i)}$ are assumed to be real, else they are complex.

If there are m independent eigenvector

Suppose $m = \dim E(r)$. So, there are m independent eigenvector $\xi^{(1)}, \ldots, \xi^{(m)}$, corresponding to the eigenvalue r.

- Then there are m solutions of (1):

$$\mathbf{y}^{(1)} = \xi^{(1)} e^{rt}, \ldots, \mathbf{y}^{(m)} = \xi^{(m)} e^{rt} \tag{2}$$

- They are linearly independent for all t.
- They extend to a fundamental set of solutions, with other $n - m$ solutions corresponding to other eigenvalues of A.

If $k = \dim E(r) \leq m - 1$

Suppose $k = \dim E(r) \leq m - 1$. The there are m_1 linearly independent eigenvector $\xi^{(1)}, \ldots, \xi^{(k)}$, with $k \leq m - 1$ corresponding to the eigenvalue r.

- Then, there are n solutions of (1):

$$\mathbf{y}^{(1)} = \xi^{(1)} e^{rt}, \ldots, \mathbf{y}^{(k)} = \xi^{(k)} e^{rt} \tag{3}$$

- They are linearly independent for all t.

Extending to m solutions

- There are algorithms that extends (3) to m solutions:

$$\left\{ \begin{array}{l} y^{(1)} = \xi^{(1)} e^{rt}, \ldots, y^{(k)} = \xi^{(k)} e^{rt}, \\ y^{(n+1)}, \ldots, y^{(m)} \end{array} \right. \tag{4}$$

which are linearly independent.

- We can say that, these m solutions described in (4) are the contributions from the eigenvalue r .
- They (4) extend to a fundamental set of solutions, with other $n-m$ solutions corresponding to other eigenvalues of A.

Complex Eigen values

If r is a complex eigenvalue of A, then so is its conjugate $\overline{r}$. Splitting the m complex solutions (4), in to real and imaginary parts, lead to $2m$ real solutions of (1), which correspond to the pair of eigenvalues r and $\overline{r}$.
In other words, the pair of conjugate complex eigenvalues r and $\overline{r}$, contribute these $2m$ solutions.

Algorithms to achieve extension (4)

To keep things simple, we would only consider the case $m = 2$. So, be r be a "double" eigenvalue of A.

- If there are two linearly independent eigen vectors $\xi^{(1)}, \xi^{(2)}$ corresponding to r, then by (2),

$$\mathrm{y}^{(1)} = \xi^{(1)} e^{rt}, \mathrm{y}^{(2)} = \xi^{(2)} e^{rt}$$

  are two solutions of (1), linearly independent, for all t.

Continued

Now suppose r is a "double" eigenvalue of A, and there is only one linearly independent eigenvector ξ for r (i. .e. $(\mathrm{A} - r\mathrm{I})\xi = 0$).

- Then $\mathrm{y}^{(1)} = \xi e^{rt}$ is a solution of (1).
- Further, the linear algebraic system

$$(\mathrm{A} - r\mathrm{I})\eta = \xi \quad \text{has a solution} \tag{5}$$

$$\text{and} \quad \mathrm{y}^{(2)} = \xi t e^{rt} + \eta e^{rt} \quad \text{is a solution of (1).} \tag{6}$$

- (*It needs a proof that (5) has a solution, which we skip.*)

- $\mathrm{y}^{(1)}, \mathrm{y}^{(2)}$ extend to a fundamental set of solutions, with other $n - m = n - 2$ solutions corresponding to other eigenvalues of A.
- It is interesting to note, by multiplying (5) by $(\mathrm{A} - r\mathrm{I})$, we have $(\mathrm{A} - r\mathrm{I})^2\eta = 0$.
- Subsequently, we ONLY consider problems with eigenvalues with multiplicity two, with only one linearly independent eigenvector.

Example 1

Find the general solution of the following system of equations:

$$\mathrm{y}' = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix} \mathrm{y} \qquad (7)$$

Computing Eigenvalues

- Eigenvalues of the coef. matrix A, are: given by

$$\begin{vmatrix} 1-r & -1 \\ 4 & -3-r \end{vmatrix} = 0 \implies (r+1)^2 = 0 \implies r = -1$$

Eigenvectors

- Eigenvectors for $r = -1$ is given by $(A - rI)\xi = 0$, which is

$$\begin{pmatrix} 1+1 & -1 \\ 4 & -3+1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{cases} 2\xi_1 - \xi_2 = 0 \\ 0 = 0 \end{cases}$$

- Taking $\xi_1 = 1$, an eigenvector of $r = -1$ is

$$\xi = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- Correspondingly, a solution of (7) is:

$$y^{(1)} = \xi e^{rt} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$$

- There is no second linearly independent eigenvector.
- So, use (6) to compute $y^{(2)}$. We proceed to solve the equation $(A - rI)\eta = \xi$

Compute η

- Write down the equation $(A - rI)\eta = \xi$ as follows:

$$\begin{pmatrix} 1+1 & -1 \\ 4 & -3+1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Longrightarrow \begin{cases} 2\eta_1 - \eta_2 = 1 \\ 0 = 0 \end{cases}$$

- Taking $\eta_1 = 1$ a choice of η is

$$\eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Answer

- By (6) another solution of (7) is

$$\mathbf{y}^{(2)} = \xi t e^{rt} + \eta e^{rt} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

- So, the general solution is $\mathbf{y} = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)}$, or

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} \right] \quad (8)$$

- Remark. While solving for η we could have taken $\eta_1 = \frac{1}{2}$ (or something else). In that case we would have

$$\eta = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

  In that case,
  - $y^{(2)}$ would be different.
  - The general solution (8), may look different. But it would be the same, by changing the constants c_1, c_2.

Example 2

Find the general solution of the following system of equations:

$$y' = \begin{pmatrix} 2 & 2 & 2 \\ 3 & 3 & -1 \\ 1 & -3 & 1 \end{pmatrix} y \qquad (9)$$

Computing Eigenvalues

Eigenvalues of the coef. matrix A, are: given by

$$\begin{vmatrix} 2-r & 2 & 2 \\ 3 & 3-r & -1 \\ 1 & -3 & 1-r \end{vmatrix} = 0$$

$$(2-r)\begin{vmatrix} 3-r & -1 \\ -3 & 1-r \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 1 & 1-r \end{vmatrix} + 2\begin{vmatrix} 3 & 3-r \\ 1 & -3 \end{vmatrix} = 0 \Longrightarrow$$

$$-r^3 + 6r^2 - 32 = 0 \Longrightarrow -(r+2)(r-4)^2 = 0$$

So, eigenvalues are: $r = 4$ with multiplicity 2. $r = -2$

Eigenvectors

Eigenvectors for $r = -2$ is given by $(A - rI)\xi = 0$:

$$\begin{pmatrix} 2+2 & 2 & 2 \\ 3 & 3+2 & -1 \\ 1 & -3 & 1+2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 4 & 2 & 2 \\ 3 & 5 & -1 \\ 1 & -3 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

TI-84 is giving clumsy output. So, I will solve it manually. Note first row is sum second and third rows. So, above system reduces to

$$\begin{pmatrix} 0 & 0 & 0 \\ 3 & 5 & -1 \\ 1 & -3 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 14 & -10 \\ 1 & -3 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{cases} 14\xi_2 - 10\xi_3 = 0 \\ \xi_1 - 3\xi_2 + 3\xi_3 = 0 \end{cases} \Longrightarrow \begin{cases} \xi_3 = 1.4\xi_2 \\ \xi_1 = 3\xi_2 - 3\xi_3 \end{cases}$$

$$\text{With} \quad \xi_2 = 10, \quad \xi_3 = 14, \quad \xi_1 = -12$$

- So, an eigenvector of $r = -2$ is:

$$\xi = \begin{pmatrix} -12 \\ 10 \\ 14 \end{pmatrix}$$

- So, a solution to (9), corresponding to $r = -2$ is $\mathrm{x}^{(1)} = \xi e^{rt}$:

$$\mathrm{y}^{(1)} = \begin{pmatrix} -12 \\ 10 \\ 14 \end{pmatrix} e^{-2t}$$

Eigenvectors for $r = 4$

- Eigenvectors for $r = 4$ is given by $(A - rI)\xi = 0$:

$$\begin{pmatrix} 2-4 & 2 & 2 \\ 3 & 3-4 & -1 \\ 1 & -3 & 1-4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} -2 & 2 & 2 \\ 3 & -1 & -1 \\ 1 & -3 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\text{Use TI84 (rref)} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Taking $\xi_2 = 1$ and eigenvector of $r = 4$ is:

$$\xi = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

- Correspondingly, a solution to (9), corresponding to $r = 2$ is $\mathrm{y}^{(2)} = \xi e^{rt}$:

$$\mathrm{y}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{4t}$$

- There is no second linearly independent eigenvector.
- So, use (6) to compute another solution $\mathrm{y}^{(3)}$. We proceed to solve the equation $(\mathrm{A} - rI)\eta = \xi$

Compute η

- Write down the equation $(A - rI)\eta = \xi$ as follows:

$$\begin{pmatrix} -2 & 2 & 2 \\ 3 & -1 & -1 \\ 1 & -3 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\text{Use TI84 (rref)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

- Taking $\eta_2 = \frac{1}{2}$ a choice of η is

$$\eta = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

Answer

- By (6) another solution of (9) is

$$\mathbf{y}^{(3)} = \xi t e^{rt} + \eta e^{rt} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{4t} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} e^{4t}$$

- So, the general solution is $\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + c_3\mathbf{y}^{(2)}$, or

$$\mathbf{x} = c_1 \begin{pmatrix} -12 \\ 10 \\ 14 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{4t}$$

$$+c_3 \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{4t} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} e^{4t} \right]$$

Example 3

Find a general solution of

$$y' = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} y$$

- First, find the eigenvalues:

$$\begin{vmatrix} 3-r & 0 & -1 \\ 0 & 2-r & 0 \\ -1 & 0 & 3-r \end{vmatrix} = 0$$

Continued

$$(r-2)(r^2-6r+8)=0$$

$$(r-2)^2(r-4)=0$$

$$r=2,2,4$$

An eigenvector and solution for $r = 2$

The eigen value $r = 2$ has multiplicity two. So, we expect two linearly independent eigen vectors.

- Eigenvetors for $r = 2$ is given by (use TI-84 "rref"):

$$\begin{pmatrix} 3-r & 0 & -1 \\ 0 & 2-r & 0 \\ -1 & 0 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \text{(use rref)}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- $$\begin{cases} \xi_1 - \xi_3 = 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

- Expect two linearly independent eigen vectors for $r = 2$. They are:
  1. Taking $\xi_2 = 1, \xi_3 = 0$, $\xi^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
  2. Likewise, taking $\xi_2 = 0, \xi_3 = 1$, $\xi^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Continued: $r = 2$

- This gives two solutions, corresponding to $r = 2$ is:

$$\begin{cases} \mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t}, \\ \mathbf{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \end{cases}$$

An eigenvector and solution for $r = 4$

- Eigenvetors for $r = 4$ is given by (use TI-84 "rref"):

$$\begin{pmatrix} 3-r & 0 & -1 \\ 0 & 2-r & 0 \\ -1 & 0 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \text{(use rref)}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- $$\begin{cases} \xi_1 = 0 \\ \xi_2 = 0 \\ 0 = 0 \end{cases}$$
- Expect two linearly independent eigen vectors for $r = 2$.

  They are: Taking $\xi_3 = 1$, $\xi^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- This gives a solution, corresponding to $r = 4$:

$$\mathrm{y}^{(3)} = \xi^{(3)} e^{rt} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{4t}$$

General Solution

- So, the general solution is:

$$y = c_1 y^{(1)} + c_2 y^{(2)} + c_3 y^{(3)}$$

$$= c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{4t} \quad (10)$$

Example 4

Solve the initial value problems

$$y' = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} y, \qquad y = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Solution

This is an extension of an example above, and the general solutions was (34):

$$y = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{4t}$$

$$= \begin{pmatrix} 0 & e^{2t} & 0 \\ e^{2t} & 0 & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Continued

Using the initial condition:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \Longrightarrow$$

$$c_1 = -1, \quad c_2 = 1, \quad c_3 = 1$$

The Answer

$$y = \begin{pmatrix} 0 & e^{2t} & 0 \\ e^{2t} & 0 & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ -e^{2t} \\ e^{4t} \end{pmatrix}$$