# Chapter 5: System of $1^{st}$-Order Linear ODE §5.6 Complex Eigenvalues

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# Complex Eigenvalues

- We continue to consider homogeneous linear systems with constant coefficients:

$$y' = Ay \quad \text{A is an } n \times n \text{ matrix with real entries} \qquad (1)$$

- In §5.5, we considered the situation when all the eigenvalues of A, were real and distinct. In this section, we consider when some of the eigen values are complex.
- As in §5.4, solutions of (1) will be denoted by

$$y^{(1)}(t), \cdots, y^{(n)}(t).$$

# Principle of superposition

- Recall the Principle of superposition
  and the converse (§5.4):
  If $y^{(1)}, \ldots, y^{(n)}$ are solution of (1), then, any constant

$$\text{linear combination} \quad y = c_1 y^{(1)} + \cdots + c_n y^{(n)} \tag{2}$$

  is also a solution of the same system (1).
- The converse is also true, if Wronskian $W \neq 0$.
- Further, if $r$ is an eigenvalue of A and
  $\xi$ is an eigenvector of $A$, corresponding to $r$, then

$$y = \xi e^{rt} \quad \text{is a solution of } (1) \tag{3}$$

# Complex eigenvalues and vectors

- Now, suppose A has a complex eigenvalue $r_1 = \lambda + i\mu$ and $\xi^{(1)}$ is an eigenvector, for $r_1$. That means

$$(A - (\lambda + i\mu)I)\xi^{(1)} = 0. \qquad (4)$$

- Apply conjugation to (4):

$$(A - (\lambda - i\mu)I)\overline{\xi^{(1)}} = 0 \qquad \text{This means :}$$

  - $r_2 = \overline{r_1} = \lambda - i\mu$ an eigenvalue of A. And,
  - $\xi^{(2)} = \overline{\xi^{(1)}}$ is an eigenvector of A, corresponding to $r_2$.

# Continued: Two conjugate complex Solutions

- Two eigen values $r_1, r_2 = \overline{r_1}$ and the corresponding eigenvalues give two solutions of (1):

$$\mathrm{y}^{(1)} = \xi^{(1)} e^{r_1 t}, \qquad \mathrm{y}^{(2)} = \xi^{(2)} e^{r_2 t} \tag{5}$$

- Write $\xi^{(1)} = \mathrm{a} + i\mathrm{b}$, where $\mathrm{a}, \mathrm{b}$ real real vectors. Then

$$\mathrm{y}^{(1)} = (\mathrm{a} + i\mathrm{b}) e^{(\lambda + i\mu)t} = (\mathrm{a} + i\mathrm{b})[e^{\lambda t}(\cos \mu t + i \sin \mu t)]$$

$$= e^{\lambda t}(\mathrm{a} \cos \mu t - \mathrm{b} \sin \mu t) + i e^{\lambda t}(\mathrm{a} \sin \mu t + \mathrm{b} \cos \mu t)$$

# Continued: Two Real Solutions

- Both real and imaginary part of $y^{(1)}$ are solutions of (1), as follows:

$$\begin{cases} u = e^{\lambda t}(a\cos\mu t - b\sin\mu t) \\ v = e^{\lambda t}(a\sin\mu t + b\cos\mu t) \end{cases} \tag{6}$$

- These real solutions u, v fit in very well as a part of a fundamental set of *n* solutions. There will be too many cases to make this statement precise, in complete details. However, we make a statement in the following frame.
- We will, usually, consider systems of 2 or 3 equations. So, following statement will suffice.

# As part of Fundamental set

**Theorem 5.6.1** Consider the homogenous linear system (1): $\mathrm{y}' = A\mathrm{y}$, where $A$ is an $n \times n$ matrix, with real entries.

- Suppose $r_1 = \lambda + i\mu$, $r_1 = \lambda - i\mu$ are two conjugate eigenvalues of A. As above, let $\xi^{(1)} = \mathrm{a} + i\mathrm{b}$ is an eigenvector of $r_1$. Accordingly, the conjugate $\xi^{(2)} = \mathrm{a} - i\mathrm{b}$ is an eigenvector of $r_2$.
- Let $\mathrm{u}, \mathrm{v}$ be as in (6).

Then there are (real) solutions $\mathrm{y}^{(3)}, \ldots, \mathrm{y}^{(n)}$ of (1), such that $\mathrm{u}, \mathrm{v}, \mathrm{y}^{(3)}, \ldots, \mathrm{y}^{(n)}$ forms a fundamental set of solutions of (1).

# Continued

Further, the solutions $\mathrm{y}^{(3)}, \ldots, \mathrm{y}^{(n)}$ are determined by the eigenvalues $r_1, r_2, r_3, \ldots, r_n$, (real or complex) and their multiplicities.

Hence, any solution x has the form (2):

$$\mathrm{x} = c_1\mathrm{u} + c_2\mathrm{v} + c_3\mathrm{y}^{(3)} + \cdots + c_n\mathrm{y}^{(n)} \tag{7}$$

# Example 1

Find the general solution (real valued) of the equation:

$$y' = \begin{pmatrix} -3 & 5 \\ -1 & 1 \end{pmatrix} y \tag{8}$$

- Eigenvalues of the coef. matrix A, are: given by

$$\begin{vmatrix} -3-r & 5 \\ -1 & 1-r \end{vmatrix} = 0 \quad r = -1+i, -1-i$$

# Eigenvectors

Analytically, eigenvectors for $r = -1 + i$ is given by $(A - rI)\xi = 0$, which is

$$\begin{pmatrix} -3-(-1+i) & 5 \\ -1 & 1-(-1+i) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The second row is $2 + i$-times the first row. It follows:

$$\begin{pmatrix} -2-i & 5 \\ -1 & 2-i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

# Continued

$$\begin{pmatrix} 0 & 0 \\ -1 & 2-i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

With $\xi_2 = 1$, an eigenvector of $r = -1 + i$ is

$$\xi^{(1)} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

# The solution

- So, the real and the imaginary part of $\xi^{(1)}$ are:

$$\mathrm{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathrm{b} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

- With $r = -1 + i$, we have $\lambda = -1, \mu = 1$. By (6),

$$\begin{cases} \mathrm{u} = e^{-t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin t\right) \\ \mathrm{v} = e^{-t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t\right) \end{cases}$$

# Continued

So, the general solution of (8)

$$\mathbf{y} = c_1\mathbf{u} + c_2\mathbf{v}$$

$$= c_1 e^{-t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin t\right) +$$

$$c_2 e^{-t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos t\right)$$

# Continued

- y =

$$c_1 e^{-t} \begin{pmatrix} 2\cos t + \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2\sin t - \cos t \\ \sin t \end{pmatrix}$$

# Example 2

Find the general solution (real valued) of the equation:

$$y' = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} y \qquad (9)$$

- Eigenvalues of the coef. matrix A, are:

$$\begin{vmatrix} 1-r & 2 & 3 \\ -2 & 1-r & 2 \\ 0 & 0 & 1-r \end{vmatrix} = 0$$

$$(1-r)\begin{vmatrix} 1-r & 2 \\ -2 & 1-r \end{vmatrix} = 0$$

So, $r = 1, 1 \pm 2i$

# Eigenvectors

- Eigenvectors for $r = 1$ is given by $(A - rI)x = 0$, which is

$$\begin{pmatrix} 1-1 & 2 & 3 \\ -2 & 1-1 & 2 \\ 0 & 0 & 1-1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Use TI-84 (rref):

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1.5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

With $\xi_3 = 2$, an eigenvector of $r = 1$ is: $\xi^{(1)} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$.

The corresponding solution $\mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^{t}$

# Eigenvectors

- Eigenvectors for $r = 1 + 2i$ is given by $(A - rI)\xi = 0$, which is

$$\begin{pmatrix} 1-(1+2i) & 2 & 3 \\ -2 & 1-(1+2i) & 2 \\ 0 & 0 & 1-(1+2i) \end{pmatrix} \xi = 0$$

$$\begin{pmatrix} -2i & 2 & 3 \\ -2 & -2i & 2 \\ 0 & 0 & -2i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$$\begin{cases} -2i\xi_1 + 2\xi_2 + 3\xi_3 = 0 \\ -2\xi_1 - 2i\xi_2 + 2\xi_3 = 0 \\ -2i\xi_3 = 0 \end{cases} \begin{cases} -2i\xi_1 + 2\xi_2 = 0 \\ -2\xi_1 - 2i\xi_2 = 0 \\ \xi_3 = 0 \end{cases} \begin{cases} -i\xi_1 + \xi_2 = 0 \\ 0 = 0 \\ \xi_3 = 0 \end{cases}$$

With $\xi_1 = 1$, an eigenvector of $r = 1 + 2i$ is:

$$\xi^{(2)} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i$$

# Solutions corresponding to $r = 1 \pm 2i$

By (6) two real solutions, corresponding to $r = 1 \pm 2i$ are:

$$\begin{cases} \mathrm{u} = e^{\lambda t}(\mathrm{a}\cos\mu t - \mathrm{b}\sin\mu t) \\ \mathrm{v} = e^{\lambda t}(\mathrm{a}\sin\mu t + \mathrm{b}\cos\mu t) \end{cases}$$

$$\begin{cases} \mathrm{u} = e^{t}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\sin 2t\right) = e^{t}\begin{pmatrix} \cos 2t \\ -\sin 2t \\ 0 \end{pmatrix} \\ \mathrm{v} = e^{t}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\sin 2t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\cos 2t\right) = e^{t}\begin{pmatrix} \sin 2t \\ \cos 2t \\ 0 \end{pmatrix} \end{cases}$$

# The general solution

Combining $\mathrm{y}^{(1)}, \mathrm{u}, \mathrm{v}$, by (7), the general solution of (9) is

$$\mathrm{x} = c_1 \mathrm{x}^{(1)} + c_2 \mathrm{u} + c_3 \mathrm{v}$$

$$= c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t + c_2 e^t \begin{pmatrix} \cos 2t \\ -\sin 2t \\ 0 \end{pmatrix} + c_3 e^t \begin{pmatrix} \sin 2t \\ \cos 2t \\ 0 \end{pmatrix}$$

titleExample 3 Solve the IVP

$$y' = \begin{pmatrix} 1 & -3 \\ 2 & 3 \end{pmatrix} y, \qquad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10}$$

- Eigenvalues of the coef. matrix A, are: given by

$$\begin{vmatrix} 1-r & -3 \\ 2 & 3-r \end{vmatrix} = 0 \Longrightarrow (1-r)(3-r)+6=0$$

So,

$$r^2-4r+9=0 \Longrightarrow r = \frac{4 \pm \sqrt{16-36}}{2}$$

So,

$$r = 2 \pm \sqrt{5}i$$

# Eigenvectors

Analytically, eigenvectors for $r = 2 + \sqrt{5}i$ is given by $(A - rI)\xi = 0$, which is

$$\begin{pmatrix} 1-(2+\sqrt{5}i) & -3 \\ 2 & 3-(2+\sqrt{5}i) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It follows:

$$\begin{pmatrix} -1-\sqrt{5}i & -3 \\ 2 & 1-\sqrt{5}i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

# Continued

$$\begin{pmatrix} 0 & 0 \\ 2 & 1-\sqrt{5}i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Taking $\xi_2 = 2$, an eigen vector is

$$\xi^{(1)} = \begin{pmatrix} -(1-\sqrt{5}i) \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix}$$

# Two Real Solutions

With $r = 2 + \sqrt{5}i$, we have $\lambda = 2, \mu = \sqrt{5}$. By (6), we have tow real solutions:

$$\begin{cases} \mathbf{u} = e^{2t}\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix} \cos\sqrt{5}t - \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix} \sin\sqrt{5}t\right) \\ \\ \mathbf{v} = e^{2t}\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix} \sin\sqrt{5}t + \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix} \cos\sqrt{5}t\right) \end{cases}$$

# Continued

We simplify:

$$\begin{cases} \mathrm{u} = e^{2t}\begin{pmatrix} -\cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t \\ 2\cos\sqrt{5}t \end{pmatrix} \\ \mathrm{v} = e^{2t}\begin{pmatrix} -\sin\sqrt{5}t + \sqrt{5}\cos\sqrt{5}t \\ 2\sin\sqrt{5}t \end{pmatrix} \end{cases}$$

# The General solution

$$\text{So, the general solutions is } \mathbf{y} = c_1\mathbf{u} + c_2\mathbf{v}$$

$$= c_1 e^{2t}\begin{pmatrix} -\cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t \\ 2\cos\sqrt{5}t \end{pmatrix}$$

$$+c_2 e^{2t}\begin{pmatrix} -\sin\sqrt{5}t + \sqrt{5}\cos\sqrt{5}t \\ 2\sin\sqrt{5}t \end{pmatrix} =$$

$$e^{2t}\begin{pmatrix} -\cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t & -\sin\sqrt{5}t + \sqrt{5}\cos\sqrt{5}t \\ 2\cos\sqrt{5}t & 2\sin\sqrt{5}t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

# Use Initial Values

Using initial conditions:

$$\begin{pmatrix} -1 & \sqrt{5} \\ 2 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So, $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{2\sqrt{5}}$

# Answer

The particular solutions is: y =

$$e^{2t}\begin{pmatrix} -\cos\sqrt{5}t-\sqrt{5}\sin\sqrt{5}t & -\sin\sqrt{5}t+\sqrt{5}\cos\sqrt{5}t \\ 2\cos\sqrt{5}t & 2\sin\sqrt{5}t \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{3}{2\sqrt{5}} \end{pmatrix}$$