Chapter 3
The Determinants
§3.1 The Determinant of a Matrix

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Goals

- We will define determinant of SQUARE matrices, inductively, using the definition of Minors and cofactors.
- We will see that determinant of triangular matrices is the product of its diagonal elements.
- Determinants are useful to compute the inverse of a matrix and solve linear systems of equations (Cramer's rule).

Overview of the definition

- Given a square matrix A, the determinant of A will be defined as a number, to be denoted by $det(A)$ or $|A|$.
- Given such a matrix A of size $n \times n$, it is possible to give a direct definition (or a formula) of $det(A)$. Unfortunately, this may be beyond the scope at this level.
- Therefore, we define inductively. That means, we first define determinant of 1×1 and 2×2 matrices. Use this to define determinant of 3×3 matrices. Then use this to define determinant of 4×4 matrices and so on we continue.

Determinant of 1×1 and 2×2 matrices

- For a 1×1 matrix $A = [a]$ define $det(A) = |A| = a$.
- Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad define \quad det(A) = |A| = ad - bc.$$

Example 3.1.1

Let

$$A = \begin{bmatrix} 2 & 17 \\ 3 & -2 \end{bmatrix} \quad then \quad det(A) = |A| = 2*(-2)-17*3 = -53$$

Example 3.1.2

Let

$$A = \begin{bmatrix} 3 & 27 \\ 1 & 9 \end{bmatrix} \quad then \quad det(A) = |A| = 3 * 9 - 1 * 27 = 0.$$

Minors of 3×3 matrices

First, we define Minors and Cofactors of 3×3 matrices.
Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the **Minor** M_{ij} of a_{ij} is defined to be the determinant of the 2×2 matrix obtained by deleting the i^{th} row and j^{th} column.

For example

$$M_{22} = \begin{vmatrix} a_{11} & & a_{13} \\ & & \\ a_{31} & & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Like wise

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}.$$

Cofactors of 3×3 matrices

Let A the 3×3 matrix as in the above frame. Then the **Cofactor** C_{ij} of a_{ij} is defined, by some sign adjustment of the minors, as follows:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For example, using the above frame

$$\begin{cases} C_{11} = (-1)^{1+1} M_{11} = M_{11} = a_{22}a_{33} - a_{23}a_{33} \\ C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -(a_{11}a_{32} - a_{12}a_{31}) \\ C_{32} = (-1)^{3+2} M_{32} = -(a_{11}a_{23} - a_{13}a_{21}). \end{cases}$$

Determinant of 3 × 3 matrices

Let A be the 3×3 matrix as above. Then the **determinant** of A is defined by

$$\det(A) = |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

This definition may be called "definition by expansion by cofactors, along the 1^{st}-row". It is possible to define the same by expansion by 2^{nd} or 3^{rd}row, which we will be discussed later.

Example 3.1.3

Let

$$A = \begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 0 \\ -2 & 1 & 1 \end{vmatrix}$$

Compute the minors M_{11}, M_{12}, M_{13}, the cofactors C_{11}, C_{12}, C_{13} and the determinant of A.

Solution:

The minors

$$M_{11} = \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix}, M_{12} = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix}, M_{13} = \begin{vmatrix} 3 & -2 \\ -2 & 1 \end{vmatrix}$$

Or

$$\begin{cases} M_{11} = -2, \\ M_{12} = 3, \\ M_{13} = -1 \end{cases}$$

Continued

So, the cofactors

$$\begin{cases} C_{11} = (-1)^{1+1}M_{11} = -2, \\ C_{12} = (-1)^{1+2}M_{12} = -3 \\ C_{13} = (-1)^{1+3}M_{13} = -1 \end{cases}$$

The determinant

$$|A| = \begin{cases} a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ = 2 * (-2) + 1 * (-3) + 1 * (-1) \\ = -8 \end{cases}$$

Example 3.1.4

Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 0 & 5 \end{pmatrix}$$

Compute the determinant of A.

Solution:

Then minors

$$M_{11} = \begin{vmatrix} -1 & 1 \\ 0 & 5 \end{vmatrix}, M_{12} = \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix}, M_{13} = \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix}$$

$$OR \begin{cases} M_{11} = -5, \\ M_{12} = 6, \\ M_{13} = 4 \end{cases} \quad Cofactors \begin{cases} C_{11} = (-1)^{1+1} M_{11} = -5, \\ C_{12} = (-1)^{1+2} M_{12} = -6, \\ C_{13} = (-1)^{1+3} M_{13} = 4 \end{cases}$$

Continued

The determinant

$$|A| = \begin{cases} a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ = 2 * (-5) + 1 * (-6) + 1 * 4 \\ = -12 \end{cases}$$

Example 3.1.5

Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix}$$

Compute the determinant of A.

Solution:

Then minors

$$M_{11} = \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix}, M_{12} = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}, M_{13} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

Or

$$\begin{cases} M_{11} = 3, \\ M_{12} = 2, \\ M_{13} = -1 \end{cases} \qquad \begin{cases} C_{11} = (-1)^{1+1} M_{11} = 3, \\ C_{12} = (-1)^{1+2} M_{12} = -2, \\ C_{13} = (-1)^{1+3} M_{13} = -1 \end{cases}$$

Continued

So, the cofactors The determianant

$$|A| = \begin{cases} a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ = 1 * 3 + 1 * (-2) + 1 * (-1) \\ = 0 \end{cases}$$

Alternative Method for 3×3 matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Form the 3×5 matrix by augmenting $1^{st}, 2^{end}$ columns to A:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & 31 & a_{32} \end{matrix}$$

Continued

Then $|A|$ can be computed as follows:

- For each of the three left to right diagonal, multiply the entries. Then add them.
- For each of the three right to left diagonal, multiply the entries. Then add them.
- $|A|$ is the difference.

Inductive process of definition

- We defined determinant of matrices size 3×3, using the determinant of 2×2 matrices.
- Now, we do the same for 4×4 matrices. This means first define minors, which would be determinant of 3×3 matrices. Then define Cofactors by adjusting the sign of the Minors.Then use the cofactors of define the determiant of the 4×4 matrix.
- The process continues, to define determinant of higher order matrices.

Minors of $n \times n$ Matrices

We assume that we know how to define determinant of $(n-1) \times (n-1)$ matrices. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{13} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \quad \textit{be a } n \times n \ \ \textit{matrix.}$$

Delete the i^{th} row and j^{th} column. The determinant of the $(n-1) \times (n-1)$ matrix obtained this way, is defined to be the (i,j)-**minor** M_{ij}. This is also loosely called the minor of a_{ij}.

Cofactors and Determinant of $n \times n$ Matrices

Let A be a $n \times n$ matrix.

- Define

$$C_{ij} = (-1)^{i+j} M_{ij} \quad \textit{which iscalled the } \textbf{cofactor} \textit{ of } a_{ij}.$$

- Define

$$det(A) = |A| = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$$

This would be called a definition
by expansion by cofactors, along first row.

Theorem 3.1.1

Theorem 3.1.1 Let A be an $n \times n$ matrix as above. Then

- $|A|$ can be computed by expanding by any row (i^{th} row):

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

- $|A|$ can ALSO be computed by expanding by any column (j^{th} column):

$$|A| = \sum_{i=1}^{n} a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

- **Proof.** Proof is needed, which we skip.

The general definition, without using induction

This may be beyond the scope at this level. However, I will attempt to explain. Let A be a $n \times n$ matrix, as above.

- **Permutation:** Any rearrangement σ of the set $\{1, 2, \ldots, n\}$ is called a permutation of $\{1, 2, \ldots, n\}$. So, σ would look like $\sigma = i_1, i_2, \ldots, i_n$ where $i_j \in \{1, 2, \ldots, n\}$ and each one appears only once.
- Permutations σ are classified into two groups: either even or odd (*outline in class*). Define **Signature**

$$sign(\sigma) = \begin{cases} 1 & if \ \sigma \ is \ even \\ -1 & if \ \sigma \ is \ odd \end{cases}$$

- We have

$$|A| = \sum_{all \ \sigma} sign(\sigma) a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n}$$

where $\sigma = j_1, j_2, \ldots, j_n$ and $\pm$ is the sign of σ.

- In fact, each term is product of n entries, such that exactly one factor comes from each row (1^{st}-index), and exactly one factor from each column (2^{nd}-index),

Definition.

Definitions. Let A be a $n \times n$ matrix.

- We say A is Upper Triangular matrix, if all entries of A below the main diagonal (left to right) are zero. In notations, if $a_{ij} = 0$ for all $i > j$.
- We say A is Lower Triangular matrix, if all entries of A above he main diagonal (left to right) are zero. In notations, if $a_{ij} = 0$ for all $i < j$.

Theorem 3.1.2

Theorem 3.1.2 Let A be a triangular matrix of order n. Then $|A|$ is product of the main-diagonal entries. Notationally,

$$|A| = a_{11}a_{22}\cdots a_{nn}.$$

Proof. The proof is easy when $n = 1, 2$. We prove it when $n = 3$. Let use assume A is lower triangular. So,

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Continued

We expand by the first row:

$$|A| = a_{11}C_{11} + 0C_{12} + 0C_{13} = a_{11}C_{11}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

For upper triangular matrices, we can prove similarly, by column expansion. For higher order matrices, we can use mathematical induction. ■

Example 1.3.6

Compute the determinant, by expansion by cofactors, of

$$A = \begin{bmatrix} x & y & z \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Solution.

- The cofactors

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 4 \\ 0 & 2 \end{vmatrix} = 8, C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = 2$$

- $$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix} = -4$$
- So, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$
$$= x * 8 + y * 2 + z * (-4) = 8x + 2y - 4z$$

Example 3.1.7.

$$Let \quad A = \begin{bmatrix} 3 & 7 & -3 & 13 \\ 0 & -7 & 2 & 17 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad Compute\ det(A).$$

Solution. This is an upper triangular matrix. So, $|A|$ is the product of the diagonal entries. So

$$|A| = 3 * (-7) * 4 * 5 = -420.$$

Example 3.1.8.

$$Let \quad A = \begin{bmatrix} 3 & 7 & -3 & 13 \\ 0 & -7 & 2 & 17 \\ 3 & 7 & 1 & 16 \\ -3 & -7 & 3 & -8 \end{bmatrix} \quad Compute\ det(A).$$

Solution. We expand, by first row. First, we compute the minors, and cofactors, of the elements in the first row.

$$M_{11} = \begin{vmatrix} -7 & 2 & 17 \\ 7 & 1 & 16 \\ -7 & 3 & -8 \end{vmatrix} = 756 \quad C_{11} = (-1)^{1+1} M_{11} = 756$$

Continued

$$M_{12} = \begin{vmatrix} 0 & 2 & 17 \\ 3 & 1 & 16 \\ -3 & 3 & -8 \end{vmatrix} = 156 \quad C_{12} = (-1)^{1+2} M_{12} = -156$$

$$M_{13} = \begin{vmatrix} 0 & -7 & 17 \\ 3 & 7 & 16 \\ -3 & -7 & -8 \end{vmatrix} = 168 \quad C_{13} = (-1)^{1+3} M_{13} = 168$$

$$M_{14} = \begin{vmatrix} 0 & -7 & 2 \\ 3 & 7 & 1 \\ -3 & -7 & 3 \end{vmatrix} = 84 \quad C_{14} = (-1)^{1+4} M_{14} = -84$$

Continued

So,

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$= 3 * 756 + 7 * (-156) - 3 * 168 + 13 * (-84) = -420$$

Example 3.1.9

$$Let \quad A = \begin{bmatrix} 3 & 7 & 1 & 21 \\ 3 & 0 & 3 & 38 \\ 3 & 7 & 1 & 16 \\ -3 & -7 & 3 & -8 \end{bmatrix} \quad Compute\ det(A).$$

Solution. We expand, by first row. First, we compute the minors, and cofactors, of the elements in the first row.

$$M_{11} = \begin{vmatrix} 0 & 3 & 38 \\ 7 & 1 & 16 \\ -7 & 3 & -8 \end{vmatrix} = 896 \quad C_{11} = (-1)^{1+1} M_{11} = 896$$

Continued

$$M_{12} = \begin{vmatrix} 3 & 3 & 38 \\ 3 & 1 & 16 \\ -3 & 3 & -8 \end{vmatrix} = 216 \quad C_{12} = (-1)^{1+2} M_{12} = -216$$

$$M_{13} = \begin{vmatrix} 3 & 0 & 38 \\ 3 & 7 & 16 \\ -3 & -7 & -8 \end{vmatrix} = 168 \quad C_{13} = (-1)^{1+3} M_{13} = 168$$

$$M_{14} = \begin{vmatrix} 3 & 0 & 3 \\ 3 & 7 & 1 \\ -3 & -7 & 3 \end{vmatrix} = 84 \quad C_{14} = (-1)^{1+4} M_{14} = -84$$

Continued

So,

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$= 3 * 896 + 7 * (-216) + 1 * 168 + 21 * (-84) = -420$$

Example 3.1.10

$$\text{Solve} \quad \begin{vmatrix} x+1 & 1 \\ -1 & x-1 \end{vmatrix} = 0$$

Solution. So,

$$(x+1)(x-1)+1=0 \quad \text{or} \quad x^2=0$$

So, $x = 0$.

Example 3.1.11

$$\textit{Solve} \quad \begin{vmatrix} x^2+1 & 2 \\ x & 1 \end{vmatrix} = 0$$

Solution. So,

$$(x^2+1)*1 - 2x = 0 \quad \textit{or} \quad (x-1)^2 = 0$$

So, $x = 1$.

Exercise 3.1.12

Let

$$A = \begin{pmatrix} 7 & \pi & -131 & 53 & 3 \\ 2 & 51 & 19 & 13 & 5 \\ 37 & -43 & 29 & -19 & 44 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\pi}{3} & 107 & -7 & 12 & -11 \end{pmatrix}$$

Compute the determinant. **Answer:** $|A| = 0$.