

Chapter 4

Vector Spaces

§4.5 Basis and Dimension

Satya Mandal

U. Kansas

Arrowtic K -Theory

Fall 2025

Goals

Discuss **two** related important concepts:

- ▶ Define **Basis** of a Vectors Space V .
- ▶ Define **Dimension** $\dim(V)$ of a Vectors Space V .

Definition: Linear Independence of infinite sets

In fact, we defined linear independence of **finite sets** S , only. Before we proceed, we define the same for infinite sets.

Definition. Suppose V is a vector space and $S \subseteq V$ is a subset (possibly infinite).

We say S is **Linearly Independent**, if any finite subset

$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subseteq S$ is linearly independent.

That means, for any finite subset $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subseteq S$ and scalars $c_1, \dots, c_n$,

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n = \mathbf{0} \implies c_1 = c_2 = \dots = c_n = 0.$$

Basis

Let V be a vector space (over $\mathbb{R}$).

A set S of vectors in V is called a **basis** of V if

1. $V = \text{Span}(S)$ and
2. S is linearly independent.

- ▶ In words, we say that *S is a basis of V* if S spans V and if S is linearly independent.
- ▶ First note, it would need a proof (i.e. it is a theorem) that any vector space has a basis. The proof requires Zorn's lemma and beyond the scope of this course.

Continued

- ▶ The definition of basis does not require that S is a finite set.
 - ▶ However, we will only deal with situations when $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a finite set.
 - ▶ If V has a finite basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, then we say that V is **finite dimensional**. Otherwise, we say that V is **infinite dimensional**.

Example 4.5.1a

The set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis of the 3-space $\mathbb{R}^3$.

Proof. We have seen, in § 4.4 that S spans $\mathbb{R}^3$ and it is linearly independent. We repeat the proof.

► Given any $(x, y, z) \in \mathbb{R}^3$ we have

$$\begin{cases} (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1). \\ \forall (x, y, z) \in \mathbb{R}^3, \quad (x, y, z) \in \text{span}(S) \\ \text{So } \mathbb{R}^3 = \text{Span}(S) \end{cases}$$

Also, S is linearly independent; because

$$a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = (0, 0, 0) \implies a = b = c = 0.$$

So, S is a basis of $\mathbb{R}^3$.

Example 4.5.1b

Similarly, a basis of the n -space $\mathbb{R}^n$ is given by the set

$$S = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$$

$$\text{where, } \begin{cases} \mathbf{e}_1 = (1, 0, 0, \dots, 0) \\ \mathbf{e}_2 = (0, 1, 0, \dots, 0) \\ \mathbf{e}_3 = (0, 0, 1, \dots, 0) \\ \dots \\ \mathbf{e}_n = (0, 0, 0, \dots, 1) \end{cases} \quad (1)$$

This one is called the **standard basis** of $\mathbb{R}^n$.

Example 4.5.2

The set $S = \{(1, -1, 0), (1, 1, 0), (1, 1, 1)\}$ is a basis of $\mathbb{R}^3$.

Proof.

- First we prove $\text{Span}(S) = \mathbb{R}^3$. Let $(x, y, z) \in \mathbb{R}^3$. We need to find a, b, c such that

$$(x, y, z) = a(1, -1, 0) + b(1, 1, 0) + c(1, 1, 1)$$

So,

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad \text{Notationally} \quad \mathbf{A}\mathbf{a} = \mathbf{v}$$

Continued

Using TI-84,
$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2 \neq 0$$

So, the above system has a solution.

Therefore $(x, y, z) \in \text{span}(S)$. So, $\text{span}(S) = \mathbb{R}^3$.

Remark. We could do the same, by long calculation.

- Now, we prove S is linearly independent. Let

$$a(1, -1, 0) + b(1, 1, 0) + c(1, 1, 1) = (0, 0, 0).$$

In the matrix form, this equation is

$$A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{where } A \text{ is as above.}$$

where A is as above. Since, $|A| = 2 \neq 0$,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, S is linearly independent.

- Since, $\text{span}(S) = \mathbb{R}^3$ and S is linearly independent, S forms a basis of $\mathbb{R}^3$.

Examples 4.5.3

- Let P_3 be a vector space of all polynomials of degree less or equal to 3. Then $S = \{1, x, x^2, x^3\}$ is a basis of P_3 .

Proof. Clearly $\text{span}(S) = P_3$. Also S is linearly independent, because

$$a1 + bx + cx^2 + dx^3 = 0 \implies a = b = c = d = 0.$$

(Why?)



Example 4.5.4

- Let $\mathbb{M}_{3,2}$ be the vector space of all 3×2 matrices. Let

$$A_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, A_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, A_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$A_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, A_{3,1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, A_{3,2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then

$$A = \{A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}, A_{3,1}, A_{3,2}\}$$

is a basis of $\mathbb{M}_{3,2}$.

Theorem 4.5.1

Theorem 4.5.1(Uniqueness of basis representation): Let V be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis of V . Then any vector $\mathbf{v} \in V$ can be written in **one and only one way** as linear combination of vectors in S .

Proof. Suppose $\mathbf{v} \in V$. Since $\text{Span}(S) = V$

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \quad \text{where} \quad a_i \in \mathbb{R}.$$

Now suppose there are two ways:

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \quad \text{and} \quad \mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$$

We will prove $a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$.

Subtracting $\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_n - b_n)\mathbf{v}_n$

Since, S is linearly independent,

$$a_1 - b_1 = 0, a_2 - b_2 = 0, \dots, a_n - b_n = 0 \text{ or}$$

$$a_1 = b_1, a_2 = b_2, \dots, a_n = b_n. \text{ The proof is complete.} \quad \blacksquare$$

Theorem 4.5.2

Theorem 4.5.2 (**Bases and cardinalities**) Let V be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis of V , containing n vectors. Then any set containing more than n vectors in V is **linearly dependent**.

Proof. Let $T = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be set of m vectors in V with $m > n$. For simplicity, assume $n = 3$ and $m = 4$. So, $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$. To prove that T is dependent, we will have to find scalars a_1, a_2, a_3, a_4 , not all zeros, such that not all zero,

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + a_4\mathbf{u}_4 = \mathbf{0} \quad \text{Equation - I}$$

We are required to show, Equation-I has **non-trivial solution** (meaning with some $a_i \neq 0$).

Continued

Since S is a basis we can write

$$\mathbf{u}_1 = c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3$$

$$\mathbf{u}_2 = c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + c_{23}\mathbf{v}_3$$

$$\mathbf{u}_3 = c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2 + c_{33}\mathbf{v}_3$$

$$\mathbf{u}_4 = c_{41}\mathbf{v}_1 + c_{42}\mathbf{v}_2 + c_{43}\mathbf{v}_3$$

We substitute these in Equation-I and re-group:

$$\begin{aligned} & (c_{11}a_1 + c_{21}a_2 + c_{31}a_3 + c_{41}a_4)\mathbf{v}_1 \\ & + (c_{12}a_1 + c_{22}a_2 + c_{32}a_3 + c_{42}a_4)\mathbf{v}_2 \\ & + (c_{13}a_1 + c_{23}a_2 + c_{33}a_3 + c_{43}a_4)\mathbf{v}_3 = \mathbf{0} \end{aligned}$$

Since $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, the coefficients of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are zero. So, we have (in the next frame):

Continued

$$c_{11}a_1 + c_{21}a_2 + c_{31}a_3 + c_{41}a_4 = 0$$

$$c_{12}a_1 + c_{22}a_2 + c_{32}a_3 + c_{42}a_4 = 0$$

$$c_{13}a_1 + c_{23}a_2 + c_{33}a_3 + c_{43}a_4 = 0$$

In matrix notation:

$$\begin{pmatrix} c_{11} & c_{21} & c_{31} & c_{41} \\ c_{12} & c_{22} & c_{32} & c_{42} \\ c_{13} & c_{23} & c_{33} & c_{43} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is a system of three **homogeneous** linear equations in four variables. (less equations than number of variable. So, the system has non-trivial (infinitely many) solutions. So, there are a_1, a_2, a_3, a_4 , not all zeros, so that Equation-I is valid. So, $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent. The proof is complete. 

Theorem 4.5.3

Suppose V is a vector space. If V has a basis with n elements then all bases have n elements.

Proof. Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $T = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ are two bases of V .

Since, the basis S has n elements, and T is linearly independent, by the theorem above m cannot be bigger than n . So, $m \leq n$.

By switching the roles of S and T , we have $n \leq m$. So, $m = n$. The proof is complete. ■

Dimension of Vector Spaces

Definition. Let V be a vector space. Suppose V has a basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ consisting of n vectors. Then we say n is the **dimension of V** and write

$$\dim(V) = n.$$

If $V = \{\mathbf{0}\}$ consists of the zero vector only, then the dimension of V is defined to be zero.
 $\dim(\{\mathbf{0}\}) = 0.$

Examples 4.5.5

We have

- ▶ From above example $\dim(\mathbb{R}^n) = n$.
- ▶ From above example $\dim(P_3) = 4$.
Similarly, $\dim(P_n) = n + 1$.
- ▶ From above example $\dim(\mathbb{M}_{3,2}) = 6$.
Similarly, $\dim(\mathbb{M}_{n,m}) = mn$.

Corollary 4.5.4: Dimensions of Subspaces

Corollary 4.5.4: Let V be a vector space and W be a subspace of V . Then

$$\dim(W) \leq \dim(V).$$

Proof. For simplicity, assume $\dim V = n < \infty$.

We give a proof by **contrapositive argument**.

Suppose $\dim W > n = \dim V$.

Then there is a basis $\mathbf{w}_1, \dots, \mathbf{w}_n, \mathbf{w}_{n+1}, \dots$ of W .

In particular, $\mathbf{w}_1, \dots, \mathbf{w}_n, \mathbf{w}_{n+1}$ is linearly independent.

Since $\dim V = n$, by Theorem 4.5.2, $\mathbf{w}_1, \dots, \mathbf{w}_n, \mathbf{w}_{n+1}$ is linearly dependent. This is a contradiction. So, $\dim W \leq \dim V$. This completes the proof. ■

Going UP

Going UP Lemma: Suppose V is a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of linearly independent elements in V . Suppose $\mathbf{v}_{k+1} \in V$ and $\mathbf{v}_{k+1} \notin \text{Span}(S)$. Then $S_{k+1} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$ is linearly independent.

Proof. Given in the end of last section. ■

Going UP

Going UP Lemma 2 Suppose V is a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of linearly independent elements in V . Then S extends to a linearly independent set

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k; \mathbf{v}_{k+1}, \dots, \mathbf{v}_n, \dots$$

The process stops when $n = \dim V$; and we get a basis.
If $\dim V = \infty$ the process continues for ever. **Proof.** Follows from Lemma UP.

Going Down

Going Down Lemma: Suppose V is a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}$ be a set of vectors and $V = \text{Span}(S)$. Suppose $\dim V = n$. Then we can **drop elements of S one-by-one**, and obtain a subset $\{\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \dots, \mathbf{v}_{i_n}\} \subseteq S$ which is a **basis** of V . **Proof.**

- ▶ S is a linearly independent then it is a basis and $N = n$.
- ▶ If not; S is linearly dependent. So, one of them is in the span of the rest. For simplicity (or relabeling them), assume $\mathbf{v}_N \in \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{N-1}\})$. Therefore $V = \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{N-1}\})$.
- ▶ Now the process continues, until we obtain a linearly independent set. Thus we obtain a basis.

Theorem 4.5.5

(**Basis Tests**): Let V be a vector space and $\dim(V) = n$.

- ▶ If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set in V (consisting of n vectors), then S is a basis of V .
- ▶ If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ spans V , then S is a basis of V

Proof.

- ▶ We first prove the first one. By going up Lemma we can extend $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ to a basis, by adding more vectors. Since $\dim V = n$, S itself must be a basis.
- ▶ To prove the latter, use going down Lemma.



Corollary 4.5.6

Let V be a vector space and $\dim(V) = n < \infty$

- ▶ Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} \subseteq V$ is a linearly independent set in V (consisting of m vectors). Then $m \leq n$ and S extends to a basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}, \dots, \mathbf{v}_n\}$ of V .
- ▶ Suppose a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} \subseteq V$ (consisting of m vectors), spans V . Then $m \geq n$ and there is a subset $T \subseteq S$, such that T is a basis of V

Proof. Follows from Going Up and Going Down Lemma. ■

Corollary 4.5.7

Let V be a vector space and Suppose
 $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} \subseteq V$ is a subset of V . Then

$$\dim(\text{Span}(S)) \leq m$$

Proof. By going down, $\text{Span}(S)$ has a basis, which is a subset of S . So, $\dim(\text{Span}(S)) \leq m$.



Example 4.5.6

$$\text{Let } W = \{(x, y, 2x + 3y) : x, y \in \mathbb{R}\}$$

Then W is a subspace of $\mathbb{R}^3$ and $\dim(W) = 2$.

Proof. Note $\mathbf{0} = (0, 0, 0) \in W$, and W is closed under addition and scalar multiplication. So, W is a subspace of $\mathbb{R}^3$. Given $(x, y, 2x + 3y) \in W$, we have

$$(x, y, 2x + 3y) = x(1, 0, 2) + y(0, 1, 3)$$

This shows $\text{span}(\{(1, 0, 2), (0, 1, 3)\}) = W$. Also $\{(1, 0, 2), (0, 1, 3)\}$ is linearly independent. So, $\{(1, 0, 2), (0, 1, 3)\}$ is a basis of W and $\dim(W) = 2$. ■

Example 4.5.7

Let

$$S = \{(1, 3, -2, 13), (-1, 2, -3, 12), (2, 1, 1, 1)\}$$

and $W = \text{span}(S)$. Prove $\dim(W) = 2$.

- ▶ **Proof.** Denote the three vectors in S by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
- ▶ Then $\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2$. Write $T = \{\mathbf{v}_1, \mathbf{v}_2\}$.
- ▶ It follows, any linear combination of vectors in S is also a linear combination of vectors in T .



$$\text{So, } W = \text{span}(S) = \text{span}(T).$$

- ▶ Also T is linearly independent. So, T is a basis and $\dim(W) = 2$.

Example 4.5.8

- **(Example)** Let $S = \{(13, 7), (-26, -14)\}$. Give a reason, why S is not a basis for $\mathbb{R}^2$?

Answer: S is linearly dependent. This is immediate because the first vector is a multiple of the second.

- **(Example)**

$$\text{Let } S = \{(5, 3, 1), (-2, 3, 1), (7, -8, 11), (\sqrt{2}, 2, \sqrt{2})\}$$

Give a reason, why S is not a basis for $\mathbb{R}^3$ where

Answer: Here $\dim(\mathbb{R}^3) = 3$. So, any basis would have 3 vectors, while S has four.

Examples 4.5.8: Continues

- **Example.** Let $S = \{1 - x, 1 - x^2, 3x^2 - 2x - 1\}$. Give a reason, why S is not a basis for P_2 ?

Answer: $\dim P_2 = 3$ and S has 3 elements. So, we have to give different reason. In fact, S is linearly dependent:

$$3x^2 - 2x - 1 = 2(1 - x) - 3(1 - x^2)$$

Examples 4.5.8: Continues

► **Example.**

$$\text{Let } S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Give a reason, why S is not a basis for $\mathbb{M}_{22}$, where

Answer: $\dim(\mathbb{M}_{22}) = 4$ and S has 3 elements.

Example 4.5.9

$$\text{Let } S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

Does S form a basis for $\mathbb{M}_{22}$, where

Answer: $\dim(\mathbb{M}_{22}) = 4$ and S has 4 elements. Further, S is linearly independent. So, S is a basis of $\mathbb{M}_{22}$. To see they are linearly independent: Let

$$a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + d \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} a + b + c + d & c + d \\ b + d & a + b + c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a = b = c = d = 0$$

Basis of subspaces

Suppose V is subspace of $\mathbb{R}^n$, spanned by a few given vectors. To find a basis of V do the following:

- ▶ Form a matrix A with these vectors, as rows.
- ▶ Then row space of A is V .
- ▶ A basis of the row space would be a basis of V , which also gives the dimension.

Example 4.5.10

Let $S = \{(3, 2, 2), (6, 5, -1), (1, 1, -1)\}$. Find a basis of $\text{span}(S)$, and $\dim(\text{span}(S))$.

Solution. Form the matrix A , with these rows.

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 6 & 5 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

Solution: We try to reduce the matrix, to a matrix **essentially** in Echelon form.

Continued

Switch first and third rows:

$$\begin{pmatrix} 1 & 1 & -1 \\ 6 & 5 & -1 \\ 3 & 2 & 2 \end{pmatrix}$$

Subtract 6 times 1st row, from 2nd and 3 times 1st row, from 3rd:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 5 \\ 0 & -1 & 5 \end{pmatrix}$$

Continued

Subtract 2^{nd} row from 3^{rd} :

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix is essentially in row Echelon form. So,

$$\begin{cases} \text{Basis of } \text{span}(S) = \{(1, 1, -1), (0, -1, 5)\} \\ \dim(\text{span}(S)) = 2 \end{cases}$$