Chapter 7
Linear Transformations
§7.1 Definitions and Introduction

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Goals

- Given two vector spaces V, W, we study the maps (i. e. functions) $T : V \to W$
  that respects the vector space structures.
- Before we proceed, in the next frame,
  we give a table of objects you have been familiar with,
  and the corresponding newer objects (or concepts)
  we did in this course.

Familiar vs. Newer

Familiar vs. Newer	
Familiar objects	**Newer Concepts**
$\mathbb{R}^n$	Vector Spaces
Lines, planes and hyper planes	Subspaces of vectors spaces
Matrices	Linear Maps

We discuss Linear Maps in this chapter.
Linear Maps would also be called Linear Transformations.

Definition of Set Theoretic Maps

- Given two sets X, Y, a function f from X to Y, written as $f : X \longrightarrow Y$, is a rule or a formula that associates, to each element $x \in X$, a unique element $f(x) \in Y$. We write $x \mapsto f(x)$.
- Such functions are also called a set theocratic maps, or simply maps.
- X is called the domain of f and Y is called the codomain of f.

Bijections

For future reference, we include the following definitions: Suppose $f : X \longrightarrow Y$ is a function from X to Y.

- We say f is a one-to-one map, if for $x_1, x_2 \in X$, $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$.
  One-to-one maps are also called injective maps.
- We say f is a onto map, if for each $y \in Y$, there is a $x \in X$ such that $f(x) = y$.
  Such "onto" maps are also called surjective maps.
- We say f is a Bijective map, if f is both injective and surjective.
  They are also called "**one-to-one and onto**" functions.

Composition

Definition: Let $f : X \longrightarrow Y$, and $g : Y \longrightarrow Z$ be two maps. The composition $gof : X \longrightarrow Z$ is the map, defined by $(gof)(x) = g(f(x))$, for all $x \in X$. We also use the notation gf for gof. Diagrammatically,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Definition: Given a set X, define $I_X : X \longrightarrow X$, by $I_X(x) = x$ for all $x \in X$. This map I_X the called the identity map, of X.

Inverse of a Map

Definition: Let $f : X \longrightarrow Y$ be a map.
A map $g : Y \longrightarrow X$ is called the inverse of f, if $gf = I_X$ and $fg = I_Y$. That means,

$$\forall\ x \in X \quad gf(x) = x, \quad \text{and} \quad \forall\ y \in Y \quad fg(y) = y.$$

Diagrammatically, following two diagrams commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "I_X"'] & Y \arrow[d, "g"] \\ & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "I_Y"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

We have the following lemma on relationships between invertible maps and bijections.

Lemma 7.1.1: Inverse and Bijections

We have the following lemma of inverses.
Lemma: Let $f : X \longrightarrow Y$ be a map. Then f has an inverse, if and only if f is **1-to-1 and onto** (i.e. bijective). **Proof.** : ($\Longrightarrow$): Suppose f has an inverse g. Then $fg = I_Y$ ad $gf = I_X$. Suppose $f(x_1) = f(x_2)$. Then

$$x_1 = g(f(x_1)) = g(f(x_2)) = x_2 \qquad So, \ f \text{ is one} - \text{to} - \text{one.}$$

Now, for $y \in Y$, we have $y = f(g(y))$.
So, f is an onto map. So, f is bijective.

Continued

($\Longleftarrow$): Suppose f is bijective. Define $g : Y \longrightarrow X$, by

$$\forall\ y \in Y \quad \text{let} \quad g(y) = x \qquad \text{if} \quad f(x) = y.$$

Then g is well defined. Also, by definition $fg = I_Y$ and $gf = I_X$. So, g is inverse of f. The proof is complete. ∎

Prelude

- Recall, a vector space V over $\mathbb{R}$ is a set V, with additional structures, namely the addition $+$ and the scalar multiplication,
  that satisfy certain conditions (ten of them) .
- Let V, W be two vector spaces over $\mathbb{R}$.
  A set theoretic map $T : V \longrightarrow W$
  is called a homomorphism,
  if T respects the vector space structures on V and W.
  We make this more precise in the next frame.

Definition

Let V, W be two vector spaces over $\mathbb{R}$ and $T : V \longrightarrow W$ be a set theocratic map. We say, T is a homomorphism if, for all vectors $\mathbf{u}, \mathbf{v} \in V$ and scalars $r \in \mathbb{R}$, the following conditions are satisfied:

$$\left\{ \begin{array}{l} T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \\ T(r\mathbf{u}) = rT(\mathbf{u}) \end{array} \right. \tag{1}$$

- Such homomorphisms of vector spaces are also called Linear maps or Linear Transformations.

Examples 7.1.1:Projection

We would consider elements of $\mathbb{R}^n$, as column vectors.

- Let $p_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the projection to the first coordinate. That means $p_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1$. Then p_1 is a homomorphism.
- Likewise, for integers $1 \leq i \leq n$, the projection $p_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ to the i^{th}-coordinate is a homomorphism.
- Further, the map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a homomorphism. This is the projection of the 3-space to the xy-plane.

Continued

Proof. We only prove the last one. Let
$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$. Then

$$T(\mathbf{u}+\mathbf{v}) = T\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix}$$

$$\text{Also,} \quad T(\mathbf{u})+T(\mathbf{v}) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix}$$

$$\text{So,} \quad T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u})+T(\mathbf{v}).$$

Continued

Also, for a scalar $c \in \mathbb{R}$, we have

$$T(r\mathbf{u}) = T\begin{pmatrix} ru_1 \\ ru_2 \\ ru_3 \end{pmatrix} = \begin{pmatrix} ru_1 \\ ru_2 \end{pmatrix}$$

$$\text{Also,} \quad rT(\mathbf{u}) = r\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ru_1 \\ ru_2 \end{pmatrix}$$

$$\text{So,} \quad T(r\mathbf{u}) = rT(\mathbf{u}).$$

Therefore, both the conditions (1) are checked. Hence, T is a homomorphism.

Example 7.1.2:Use homogeneous linear Polynomials

We can use homogeneous linear polynomials to construct examples of Linear maps. Here is one:
Define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, as follows

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ x-y+z \end{pmatrix}.$$

Then T is a homomorphism.

Remark. In matrix notations $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Continued

Proof. Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$. Then

$$T(\mathbf{u}+\mathbf{v}) = T\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix}$$

$$= \begin{pmatrix} (u_1+v_1)+2(u_2+v_2)+3(u_3+v_3) \\ (u_1+v_1)-(u_2+v_2)+(u_3+v_3) \end{pmatrix}$$

$$= \begin{pmatrix} u_1+2u_2+3u_3 \\ u_1-u_2+u_3 \end{pmatrix} + \begin{pmatrix} v_1+2v_2+3v_3 \\ v_1-v_2+v_3 \end{pmatrix}$$

$$= T(\mathbf{u}) + T(\mathbf{v})$$

So, the first condition of (1) is checked.

Continued

For a scalar $r \in \mathbb{R}$, we have

$$T(r\mathbf{u}) = T\begin{pmatrix} ru_1 \\ ru_2 \\ ru_3 \end{pmatrix}$$

$$= \begin{pmatrix} ru_1 + 2ru_2 + 3ru_3 \\ ru_1 - ru_2 + ru_3 \end{pmatrix} = r\begin{pmatrix} u_1 + 2u_2 + 3u_3 \\ u_1 - u_2 + u_3 \end{pmatrix}$$

$$= rT(\mathbf{u})$$

So, the second condition of (1) is checked.
Therefore, T is a homomorphism.

Example 71.3:Use Matrices

The approach in Example 7.1.2 can be generalized, using matrices.
Suppose A is a $m \times n$-matrix. Define

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{by} \quad T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

Then T is a linear transformation.
(*This is probably the most relevant example, for us.*)

Continued

Proof. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $r \in \mathbb{R}$, we have

$$\begin{cases} T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v}) \\ T(r\mathbf{u}) = A(r\mathbf{u}) = r(A\mathbf{u}) = rT(\mathbf{u}) \end{cases}$$

So, both the conditions of (1) are satisfied.
Therefore, T is a homomorphism.

Example 7.1.4:Inclusions

Usual inclusion of vector spaces are homomorphisms.

Here is one:

Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$, as follows

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix}. \quad \text{Then } T \text{ is a homomorphism.}$$

Proof. Exercise.

Example 7.1.5:Matrices to Matrices

I commented that the vector space $\mathbb{M}_{m\times n}(\mathbb{R})$ of all matrices of size $m \times n$ is "same as" the vector space $\mathbb{R}^{mn}$. But one can construct some interesting example. Here is one:
Define $T : \mathbb{M}_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{M}_{4\times 3}(\mathbb{R})$, as follows

$$T\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then T is a homomorphism. **Proof.** Exercise. (*Note the use of* 0*s*)

Example 7.1.6:Matrices to Matrices

Here is another one:
Define $T : \mathbb{M}_{4\times 3}(\mathbb{R}) \longrightarrow \mathbb{M}_{3\times 3}(\mathbb{R})$, as follows

$$T \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then T is a homomorphism. **Proof.** Exercise.

Example 7.1.7:Trace of a Matrix

Here is another one:
Define $T : \mathbb{M}_{3\times 3}(\mathbb{R}) \longrightarrow \mathbb{R}$, as follows

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} + a_{22} + a_{33}$$

Then T is a homomorphism.
This example is called the "trace" of the matrix.
More generally, one can define the "trace"

$$T : \mathbb{M}_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R} \quad \text{by} \quad T(A) = \sum_{i=1}^{n} a_{ii} = \sum \text{diagonal entries.}$$

Proof. Exercise.

Non-Example 7.1.8:Use Linear polynomials

We modify one of the above examples: Define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, as follows

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z + 1 \\ x - y + z \end{pmatrix}.$$

Then T is not a homomorphism.
Proof. The presence of the constant term 1 is the problem. Now, one can give many proofs. For example,

Continued

$$T\left(2\begin{pmatrix}1\\0\\0\end{pmatrix}\right)=T\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$$

$$2T\begin{pmatrix}1\\0\\0\end{pmatrix}=2\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}$$

$$\text{So } T\left(2\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\neq 2T\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

So, 2^{nd} condition of (1) fails. So, T is not a homomorphism.

Non-Example 7.1.9:Use non-Linear polynomials

Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, as follows

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ x - y \end{pmatrix}.$$

Then T is not a homomorphism. **Proof.** In fact, both conditions (1) would fail, because $x^2 + y^2$ is not linear. For example,

$$T\left(2\begin{pmatrix} x \\ y \end{pmatrix}\right) = T\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} 4x^2 + 4y^2 \\ 2x - 2y \end{pmatrix}$$

Continued

$$2T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2(x^2+y^2) \\ 2(x-y) \end{pmatrix} \quad \text{Therefore,}$$

$$T\left(2\begin{pmatrix} x \\ y \end{pmatrix}\right) \neq 2T\begin{pmatrix} x \\ y \end{pmatrix}. \quad 2^{\text{nd}} \text{ condition of (1) fails.}$$

Non-Example L.1.10:Determinant

The determinant function $\det : \mathbb{M}_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}$ is not a homomorphism of vector spaces.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\det(2A) = \det\left(2\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \det\begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} = 4(ad-bc)$$

$$2\det(A) = 2\det(\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2(ad-bc)$$

So, $\det(2A) \neq 2\det(A)$. So, the 2^{nd} condition of (1) fails. So, det-function is not a homomorphism.

Exercises 1

1. Let V be an inner product space and $\mathbf{u} \in V$, with $\mathbf{u} \neq \mathbf{0}$. For $\mathbf{x} \in V$, define

$$T(\mathbf{x}) = Proj_{\mathbf{u}}\mathbf{x} = \frac{\langle \mathbf{u}, \mathbf{x} \rangle}{\| \mathbf{u} \|}\mathbf{u}$$

Prove that $T \longrightarrow T$ is a homomorphism.