Levent Alpöge's AI counter example to the Jacobian Conjecture

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Abstract

We give a hand computed exposition of the counter example to the Jacobian conjecture, due to Levent Alpöge.

The purpose of this writeup is to provide a hand computed complete proof of the counter example to the Jacobian conjecture provided by Levent Alpöge [A26]. The problem has a wider interest outside the mathematics community. I was inspired by requests from some of my friends (statisticians, economists and others) to write such an exposition. We claim no originality.

1 The Counter Example

Levent Alpöge announced an explicit , hand-checkable counterexample [A26] in three dimensions that he and others discovered using the AI model Claude Fable 5. I maintain a low profile in the social media. At the time of this writing, the announcement [A26] may be too old. A colleague called me about three days ago that Alpöge gave a hand checkable counter example of Jacobian conjecture. After I drooped the phone, it did not take more than three minutes for me to find out the example from the net. The example is as follows:

$$\left\{\begin{array}{l} F_1 = (1+XY)^3 Z + Y^2 (1+XY)(4+3XY) \\ F_2 = Y + 3X(1+XY)^2 Z + 3XY^2(4+3XY) \\ F_3 = 2X - 3X^2Y - X^3 Z \end{array}\right. \tag{1}$$

It is claimed, which is a theorem now, that this provides a counter example to the Jacobian conjecture, as follows.

Theorem 1.1 (Alpöge)**.** Define the function

$$\mathbb{C}^3 \xrightarrow{\quad F \quad} \mathbb{C}^3 \quad \text{by} \quad F(x,y,z) = \left(F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\right)$$

Then F is not an injective map. However, the Jacobian determinant

$$|J| = \begin{vmatrix} \frac{\partial F_1}{\partial X} & \frac{\partial F_2}{\partial X} & \frac{\partial F_3}{\partial X} \\ \frac{\partial F_1}{\partial Y} & \frac{\partial F_2}{\partial Y} & \frac{\partial F_3}{\partial Y} \\ \frac{\partial F_1}{\partial Z} & \frac{\partial F_2}{\partial Z} & \frac{\partial F_3}{\partial Z} \end{vmatrix} = -2 \neq 0 \tag{2}$$

Consequently, the Jacobian conjecture fails for this example.

Proof. It is given in [A26], and is easy to check that

$$F\left(0,0,-\frac{1}{4}\right) = F\left(1,-\frac{3}{2},\frac{13}{2}\right) = F\left(-1,\frac{3}{2},\frac{13}{2}\right) = \left(-\frac{1}{4},0,0\right)$$

So, F is not injective. The equality (2) is of course hand checkable, while it will be laborious. ■

The purpose of the rest of this writeup is to write down a proof the equality (2).

1.1 Computation of the Jacobian

In this section we give a proof of the equality (2).

Proof. For typographical reasons, we write down three columns of the Jacobian matrix J, one below the other

$$J = \begin{array}{l} \frac{\partial F_1}{\partial X} = 3(1+XY)^2YZ + Y^2\left(Y(4+3XY) + 3(1+XY)Y\right) \\ \frac{\partial F_1}{\partial Y} = 3(1+XY)^2XZ + 2Y(1+XY)(4+3XY) + Y^2X(4+3XY) + 3XY^2(1+XY) \\ \frac{\partial F_1}{\partial Z} = (1+XY)^3 \\ \hline \frac{\partial F_2}{\partial X} = 3(1+XY)^2Z + 6XY(1+XY)Z + 3Y^2(4+3XY) + 9XY^3 \\ \frac{\partial F_2}{\partial Y} = 1 + 6X^2(1+XY)Z + 6XY(4+3XY) + 9X^2Y^2 \\ \frac{\partial F_2}{\partial Z} = 3X(1+XY)^2 \\ \hline \frac{\partial F_3}{\partial X} = 2 - 6XY - 3X^2Z \\ \frac{\partial F_3}{\partial Y} = -3X^2 \\ \frac{\partial F_3}{\partial Z} = -X^3 \end{array}$$

By simplification, we have

$$J = \begin{array}{l} \begin{pmatrix} 3X^2Y^3Z + 6XY^4 + 6XY^2Z + 7Y^3 + 3YZ \\ 3X^3Y^2Z + 12X^2Y^3 + 6X^2YZ + 21XY^2 + 3XZ + 8Y \\ (1+XY)^3 \end{pmatrix} \\ \hline \begin{pmatrix} 9X^2Y^2Z + 12XYZ + 18XY^3 + 12Y^2 + 3Z \\ 6X^3YZ + 27X^2Y^2 + 6X^2Z + 24XY + 1 \\ 3X(1+XY)^2 \end{pmatrix} \\ \hline \begin{pmatrix} 2 - 6XY - 3X^2Z \\ -3X^2 \\ -X^3 \end{pmatrix} \end{array}$$

We compute the determinant $|J|$ by expanding along the last row of the matrix. Subsequently, M_{ij} denotes the ij-minor of J. We have

$$-X^3 M_{33} = -X^3 \left\{ \begin{array}{l} \left\{ \begin{array}{l} (3X^2Y^3Z + 6XY^4 + 6XY^2Z + 7Y^3 + 3YZ) \\ \times (6X^3YZ + 27X^2Y^2 + 6X^2Z + 24XY + 1) \end{array} \right. \\ - \\ \left\{ \begin{array}{l} (3X^3Y^2Z + 12X^2Y^3 + 6X^2YZ + 21XY^2 + 3XZ + 8Y) \\ \times (9X^2Y^2Z + 12XYZ + 18XY^3 + 12Y^2 + 3Z) \end{array} \right. \end{array} \right.$$

So

$$\begin{array}{l} -X^3 M_{33} = \\ -X^3 \left\{ \begin{array}{l} \left\{ \begin{array}{l} (18X^5Y^4Z^2 + 36X^4Y^5Z + 36X^4Y^3Z^2 + 42X^3Y^4Z + 18X^3Y^2Z^2) \\ + (81X^4Y^5Z + 162X^3Y^6 + 162X^3Y^4Z + 189X^2Y^5 + 81X^2Y^3Z) \\ + (18X^4Y^3Z^2 + 36X^3Y^4Z + 36X^3Y^2Z^2 + 42X^2Y^3Z + 18X^2YZ^2) \\ + (72X^3Y^4Z + 144X^2Y^5 + 144X^2Y^3Z + 168XY^4 + 72XY^2Z) \\ + (3X^2Y^3Z + 6XY^4 + 6XY^2Z + 7Y^3 + 3YZ) \end{array} \right. \\ - \\ \left\{ \begin{array}{l} (27X^5Y^4Z^2 + 108X^4Y^5Z + 54X^4Y^3Z^2 + 189X^3Y^4Z + 27X^3Y^2Z^2 + 72X^2Y^3Z) \\ (36X^4Y^3Z^2 + 144X^3Y^4Z + 72X^3Y^2Z^2 + 252X^2Y^3Z + 36X^2YZ^2 + 96XY^2Z) \\ (54X^4Y^5Z + 216X^3Y^6 + 108X^3Y^4Z + 378X^2Y^5 + 54X^2Y^3Z + 144XY^4) \\ (36X^3Y^4Z + 144X^2Y^5 + 72X^2Y^3Z + 252XY^4 + 36XY^2Z + 96Y^3) \\ (9X^3Y^2Z^2 + 36X^2Y^3Z + 18X^2YZ^2 + 63XY^2Z + 9XZ^2 + 24YZ) \end{array} \right. \end{array} \right. \end{array}$$

$$= -X^3 \left\{ \begin{array}{l} \left\{ \begin{array}{l} (18X^5Y^4Z^2 + 36X^4Y^5Z + 36X^4Y^3Z^2 + 42X^3Y^4Z + 18X^3Y^2Z^2) \\ + (81X^4Y^5Z + 162X^3Y^6 + 162X^3Y^4Z + 189X^2Y^5 + 81X^2Y^3Z) \\ + (18X^4Y^3Z^2 + 36X^3Y^4Z + 36X^3Y^2Z^2 + 42X^2Y^3Z + 18X^2YZ^2) \\ + (72X^3Y^4Z + 144X^2Y^5 + 144X^2Y^3Z + 168XY^4 + 72XY^2Z) \\ + (3X^2Y^3Z + 6XY^4 + 6XY^2Z + 7Y^3 + 3YZ) \end{array} \right. \\ - \\ \left\{ \begin{array}{l} (27X^5Y^4Z^2 + 108X^4Y^5Z + 54X^4Y^3Z^2 + 189X^3Y^4Z + 27X^3Y^2Z^2) + 72X^2Y^3Z \\ (36X^4Y^3Z^2 + 144X^3Y^4Z + 72X^3Y^2Z^2 + 252X^2Y^3Z + 36X^2YZ^2 + 96XY^2Z) \\ (54X^4Y^5Z + 216X^3Y^6 + 108X^3Y^4Z + 378X^2Y^5 + 54X^2Y^3Z + 144XY^4) \\ (36X^3Y^4Z + 144X^2Y^5 + 72X^2Y^3Z + 252XY^4 + 36XY^2Z + 96Y^3) \\ (\mathbf{9X^3Y^2Z^2} + 36X^2Y^3Z + 18X^2YZ^2 + 63XY^2Z + 9XZ^2 + 24YZ) \end{array} \right. \end{array} \right.$$

$$= -X^3 \left\{ \begin{array}{l} \left\{ \begin{array}{l} -9X^5Y^4Z^2 + 9X^4Y^5Z - 36X^4Y^3Z^2 - 165X^3Y^4Z - 54X^3Y^2Z^2 \\ + (162X^3Y^6 + 189X^2Y^5 + 81X^2Y^3Z) \\ + (+42X^2Y^3Z + 18X^2YZ^2) \\ + (144X^2Y^5 + 144X^2Y^3Z + 168XY^4 + 72XY^2Z) \\ + (3X^2Y^3Z + 6XY^4 + 6XY^2Z + 7Y^3 + 3YZ) \end{array} \right. \\ - \\ \left\{ \begin{array}{l} (72X^2Y^3Z + 252X^2Y^3Z + 36X^2YZ^2 + 96XY^2Z) \\ (54X^4Y^5Z + 216X^3Y^6 + 378X^2Y^5 + 54X^2Y^3Z + 144XY^4) \\ (144X^2Y^5 + 72X^2Y^3Z + 252XY^4 + 36XY^2Z + 96Y^3) \\ (+36X^2Y^3Z + 18X^2YZ^2 + 63XY^2Z + 9XZ^2 + 24YZ) \end{array} \right. \end{array} \right.$$

$$= -X^3 \left\{ \begin{array}{l} \left\{ \begin{array}{l} -9X^5Y^4Z^2 + 9X^4Y^5Z - 36X^4Y^3Z^2 - 165X^3Y^4Z - 54X^3Y^2Z^2 \\ -54X^3Y^6 - 189X^2Y^5 - 216X^2Y^3Z - 18X^2YZ^2 \\ 168XY^4 + 72XY^2Z \\ +(+6XY^4 + 6XY^2Z + 7Y^3 + 3YZ) \end{array} \right. \\ - \\ \left\{ \begin{array}{l} (96XY^2Z54X^4Y^5Z + 144XY^4 + 252XY^4 + 36XY^2Z + 96Y^3) \\ (+18X^2YZ^2 + 63XY^2Z + 9XZ^2 + 24YZ) \end{array} \right. \end{array} \right.$$

$$= -X^3 \left\{ \begin{array}{l} \left\{ \begin{array}{l} -9X^5Y^4Z^2 + 9X^4Y^5Z - 36X^4Y^3Z^2 - 165X^3Y^4Z - 54X^3Y^2Z^2 \\ -54X^3Y^6 - 189X^2Y^5 - 216X^2Y^3Z - 18X^2YZ^2 \\ -222XY^4 - 117XY^2Z + 7Y^3 + 3YZ \end{array} \right. \\ - \\ \left\{ \ 54X^4Y^5Z + 96Y^3 + 18X^2YZ^2 + +9XZ^2 + 24YZ \right. \end{array} \right.$$

So ,

$$-X^3M_{33} = -X^3 \left\{ \begin{array}{l} -9X^5Y^4Z^2 - 45X^4Y^5Z - 36X^4Y^3Z^2 - 165X^3Y^4Z - 54X^3Y^2Z^2 \\ -54X^3Y^6 - 189X^2Y^5 - 216X^2Y^3Z - 36X^2YZ^2 \\ -222XY^4 - 117XY^2Z - 9XZ^2 - 21YZ - 89Y^3 \end{array} \right.$$

Finally

$$-X^3M_{33} = \left\{ \begin{array}{l} 9X^8Y^4Z^2 + 45X^7Y^5Z + 36X^7Y^3Z^2 + 165X^6Y^4Z + 54X^6Y^2Z^2 \\ +54X^6Y^6 + 189X^5Y^5 + 216X^5Y^3Z + 36X^5YZ^2 \\ +222X^4Y^4 + 117X^4Y^2Z + 9X^4Z^2 + 21X^3YZ + 89X^3Y^3 \end{array} \right. \tag{3}$$

Now we compute M_{32}.

$$M_{32} = \left| \begin{pmatrix} 3X^2Y^3Z + 6XY^4 + 6XY^2Z + 7Y^3 + 3YZ & 2 - 6XY - 3X^2Z \\ 3X^3Y^2Z + 12X^2Y^3 + 6X^2YZ + 21XY^2 + 3XZ + 8Y & -3X^2 \end{pmatrix} \right|$$

$$= \left\{ \begin{array}{l} -3X^2\left(3X^2Y^3Z + 6XY^4 + 6XY^2Z + 7Y^3 + 3YZ\right) \\ - \\ \left\{ \begin{array}{l} (3X^3Y^2Z + 12X^2Y^3 + 6X^2YZ + 21XY^2 + 3XZ + 8Y) \\ \times(2 - 6XY - 3X^2Z) \end{array} \right. \end{array} \right.$$

$$= \left\{ \begin{array}{l} (-9X^4Y^3Z - 18X^3Y^4 - 18X^3Y^2Z - 21X^2Y^3 - 9X^2YZ) \\ - \\ \left\{ \begin{array}{l} (6X^3Y^2Z + 24X^2Y^3 + 12X^2YZ + 42XY^2 + 6XZ + 16Y) \\ -(18X^4Y^3Z + 72X^3Y^4 + 36X^3Y^2Z + 126X^2Y^3 + 18X^2YZ + 48XY^2) \\ -(9X^5Y^2Z^2 + 36X^4Y^3Z + 18X^4YZ^2 + 63X^3Y^2Z + 9X^3Z^2 + 24X^2YZ) \end{array} \right. \end{array} \right.$$

$$= \left\{ \begin{array}{l} (-9X^4Y^3Z - 18X^3Y^4 - 18X^3Y^2Z - 21X^2Y^3 - 9X^2YZ) \\ + \\ \left\{ \begin{array}{l} (-6X^3Y^2Z - 24X^2Y^3 - 12X^2YZ - 42XY^2 - 6XZ - 16Y) \\ +(18X^4Y^3Z + 72X^3Y^4 + 36X^3Y^2Z + 126X^2Y^3 + 18X^2YZ + 48XY^2) \\ +(\mathbf{9X^5Y^2Z^2} + 36X^4Y^3Z + 18X^4YZ^2 + 63X^3Y^2Z + 9X^3Z^2 + 24X^2YZ) \end{array} \right. \end{array} \right.$$

Therfore

$$\left\{\begin{array}{l} -\left(3X(1+XY)^2\right) M_{32} = \\ \left\{\begin{array}{l} -\left(3X(1+XY)^2\right) \times \\ 9X^5Y^2Z^2 + 18X^4YZ^2 + 45X^4Y^3Z + 54X^3Y^4 + 75X^3Y^2Z + 9X^3Z^2 \\ +81X^2Y^3 + 21X^2YZ + 6XY^2 - 6XZ - 16Y \end{array}\right. \end{array}\right. \tag{4}$$

Now, we compute M_{31}. We have

$$M_{31} = \left|\begin{pmatrix} 9X^2Y^2Z + 12XYZ + 18XY^3 + 12Y^2 + 3Z & 2 - 6XY - 3X^2Z \\ 6X^3YZ + 27X^2Y^2 + 6X^2Z + 24XY + 1 & -3X^2 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} 9X^2Y^2Z + 12XYZ + 18XY^3 + 12Y^2 + 3Z & 2 - 6XY - 3X^2Z \\ 6X^3YZ + 27X^2Y^2 + 6X^2Z + 24XY + 1 & -3X^2 \end{pmatrix}\right|$$

$$= \left\{\begin{array}{l} -3X^2\left(9X^2Y^2Z + 12XYZ + 18XY^3 + 12Y^2 + 3Z\right) \\ - \\ \left\{\begin{array}{l} 6X^3YZ + 27X^2Y^2 + 6X^2Z + 24XY + 1 \\ \times\left(2 - 6XY - 3X^2Z\right) \end{array}\right. \end{array}\right.$$

$$= \left\{\begin{array}{l} \left(-27X^4Y^2Z - 36X^3YZ - 54X^3Y^3 - 36X^2Y^2 - 9X^2Z\right) \\ - \\ \left\{\begin{array}{l} 12X^3YZ + 54X^2Y^2 + 12X^2Z + 48XY + 2 \\ -36X^4Y^2Z - 162X^3Y^3 - 36X^3YZ - 144X^2Y^2 - 6XY \\ -18X^5YZ^2 - 81X^4Y^2Z - 18X^4Z^2 - 72X^3YZ - 3X^2Z \end{array}\right. \end{array}\right.$$

$$= \left\{\begin{array}{l} 18X^5YZ^2 + 90X^4Y^2Z + 18X^4Z^2 + 60X^3YZ + 108X^3Y^3 \\ 54X^2Y^2 - 18X^2Z - 42XY - 2 \end{array}\right.$$

Therefore

$$\left\{\begin{array}{l} (1+XY)^3 M_{31} = \\ (1+XY)^3 \times \\ \left\{\begin{array}{l} 18X^5YZ^2 + 90X^4Y^2Z + 18X^4Z^2 + 60X^3YZ + 108X^3Y^3 \\ 54X^2Y^2 - 18X^2Z - 42XY - 2 \end{array}\right. \end{array}\right. \tag{5}$$

Combining (5, 4, 3) the Jacobian determianant

$$|J| = \left\{\begin{array}{l} (1+XY)^3 \left\{\begin{array}{l} 18X^5YZ^2 + 90X^4Y^2Z + 18X^4Z^2 + 60X^3YZ + 108X^3Y^3 \\ 54X^2Y^2 - 18X^2Z - 42XY - 2 \end{array}\right. \\ + \\ \left\{\begin{array}{l} -\left(3X(1+XY)^2\right) \times \\ 9X^5Y^2Z^2 + 18X^4YZ^2 + 45X^4Y^3Z + 54X^3Y^4 + 75X^3Y^2Z + 9X^3Z^2 \\ +81X^2Y^3 + 21X^2YZ + 6XY^2 - 6XZ - 16Y \end{array}\right. \\ + \\ \left\{\begin{array}{l} 9X^8Y^4Z^2 + 45X^7Y^5Z + 36X^7Y^3Z^2 + 165X^6Y^4Z + 54X^6Y^2Z^2 \\ +54X^6Y^6 + 189X^5Y^5 + 216X^5Y^3Z + 36X^5YZ^2 \\ +222X^4Y^4 + 117X^4Y^2Z - 9X^4Z^2 + 21X^3YZ + 89X^3Y^3 \end{array}\right. \end{array}\right.$$

So,

$$|J| = \left\{ \begin{array}{l} (1+XY)^2 \left\{ \begin{array}{l} 18X^5YZ^2 + 90X^4Y^2Z + 18X^4Z^2 + 60X^3YZ + 108X^3Y^3 \\ +54X^2Y^2 - 18X^2Z - 42XY - 2 \\ 18X^6Y^2Z^2 + 90X^5Y^3Z + 18X^5YZ^2 + 60X^4Y^2Z + 108X^4Y^4 \\ +54X^3Y^3 - 18X^3YZ - 42X^2Y^2 - 2XY \end{array}\right. \\ + \\ \left\{ \begin{array}{l} -(1+XY)^2 \times \\ 27X^6Y^2Z^2 + 54X^5YZ^2 + 135X^5Y^3Z + 162X^4Y^4 + 225X^4Y^2Z + 27X^4Z^2 \\ +243X^3Y^3 + 63X^3YZ + 18X^2Y^2 - 18X^2Z - 48XY \end{array}\right. \\ + \\ \left\{ \begin{array}{l} 9X^8Y^4Z^2 + 45X^7Y^5Z + 36X^7Y^3Z^2 + 165X^6Y^4Z + 54X^6Y^2Z^2 \\ +54X^6Y^6 + 189X^5Y^5 + 216X^5Y^3Z + 36X^5YZ^2 \\ +222X^4Y^4 + 117X^4Y^2Z - 9X^4Z^2 + 21X^3YZ + 89X^3Y^3 \end{array}\right. \end{array}\right.$$

$$= \left\{ \begin{array}{l} (1+XY)^2 \left\{ \begin{array}{l} -9X^6Y^2Z^2 - 45X^5Y^3Z - 18X^5YZ^2 - 54X^4Y^4 - 75X^4Y^2Z - 9X^4Z^2 \\ -21X^3YZ - 81X^3Y^3 - 6X^2Y^2 + 4XY - 2 \end{array}\right. \\ + \\ \left\{ \begin{array}{l} 9X^8Y^4Z^2 + 45X^7Y^5Z + 36X^7Y^3Z^2 + 165X^6Y^4Z + 54X^6Y^2Z^2 \\ +54X^6Y^6 + 189X^5Y^5 + 216X^5Y^3Z + 36X^5YZ^2 \\ +222X^4Y^4 + 117X^4Y^2Z - 9X^4Z^2 + 21X^3YZ + 89X^3Y^3 \end{array}\right. \end{array}\right.$$

$$= \left\{ \begin{array}{l} \left\{ \begin{array}{l} -9X^6Y^2Z^2 - 45X^5Y^3Z - 18X^5YZ^2 - 54X^4Y^4 - 75X^4Y^2Z - 9X^4Z^2 \\ -21X^3YZ - 81X^3Y^3 - 6X^2Y^2 + 4XY - 2 \\ -18X^7Y^3Z^2 - 90X^6Y^4Z - 36X^6Y^2Z^2 - 108X^5Y^5 - 150X^5Y^3Z - 18X^5YZ^2 \\ -42X^4Y^2Z - 162X^4Y^4 - 12X^3Y^3 + 8X^2Y^2 - 4XY \\ -9X^8Y^4Z^2 - 45X^7Y^5Z - 18X^7Y^3Z^2 - 54X^6Y^6 - 75X^6Y^4Z - 9X^6Y^2Z^2 \\ -21X^5Y^3Z - 81X^5Y^5 - 6X^4Y^4 + 4X^3Y^3 - 2X^2Y^2 \end{array}\right. \\ + \\ \left\{ \begin{array}{l} 9X^8Y^4Z^2 + 45X^7Y^5Z + 36X^7Y^3Z^2 + 165X^6Y^4Z + 54X^6Y^2Z^2 \\ +54X^6Y^6 + 189X^5Y^5 + 216X^5Y^3Z + 36X^5YZ^2 \\ +222X^4Y^4 + 117X^4Y^2Z - 9X^4Z^2 + 21X^3YZ + 89X^3Y^3 \end{array}\right. \end{array}\right.$$

$$= \left\{ \begin{array}{l} \left\{ \begin{array}{l} -9X^6Y^2Z^2 - 45X^5Y^3Z - 18X^5YZ^2 - 54X^4Y^4 - 75X^4Y^2Z - 9X^4Z^2 \\ -21X^3YZ - 81X^3Y^3 - 6X^2Y^2 + 4XY - 2 \\ -18X^7Y^3Z^2 - 90X^6Y^4Z - 36X^6Y^2Z^2 - 108X^5Y^5 - 150X^5Y^3Z - 18X^5YZ^2 \\ -42X^4Y^2Z - 162X^4Y^4 - 12X^3Y^3 + 8X^2Y^2 - 4XY \\ -9X^8Y^4Z^2 - 45X^7Y^5Z - 18X^7Y^3Z^2 - 54X^6Y^6 - 75X^6Y^4Z - 9X^6Y^2Z^2 \\ -21X^5Y^3Z - 81X^5Y^5 - 6X^4Y^4 + 4X^3Y^3 - 2X^2Y^2 \end{array}\right. \\ + \\ \left\{ \begin{array}{l} 9X^8Y^4Z^2 + 45X^7Y^5Z + 36X^7Y^3Z^2 + 165X^6Y^4Z + 54X^6Y^2Z^2 \\ +54X^6Y^6 + 189X^5Y^5 + 216X^5Y^3Z + 36X^5YZ^2 \\ +222X^4Y^4 + 117X^4Y^2Z - 9X^4Z^2 + 21X^3YZ + 89X^3Y^3 \end{array}\right. \end{array}\right.$$

Therefore

$$|J| = \left\{ \begin{array}{l} \left\{ \begin{array}{l} -54X^6Y^2Z^2 - 216X^5Y^3Z - 36X^5YZ^2 - 222X^4Y^4 - 117X^4Y^2Z - 9X^4Z^2 \\ -21X^3YZ - 89X^3Y^3 - 2 \\ -36X^7Y^3Z^2 - 165X^6Y^4Z - 189X^5Y^5 \\ -9X^8Y^4Z^2 - 45X^7Y^5Z - 54X^6Y^6 \end{array} \right. \\ + \\ \left\{ \begin{array}{l} 9X^8Y^4Z^2 + 45X^7Y^5Z + 36X^7Y^3Z^2 + 165X^6Y^4Z + 54X^6Y^2Z^2 \\ +54X^6Y^6 + 189X^5Y^5 + 216X^5Y^3Z + 36X^5YZ^2 \\ +222X^4Y^4 + 117X^4Y^2Z + 9X^4Z^2 + 21X^3YZ + 89X^3Y^3 \end{array} \right. \end{array} \right.$$

$$= \left\{ \begin{array}{l} \left\{ \begin{array}{l} -9X^8Y^4Z^2 - 45X^7Y^5Z - 36X^7Y^3Z^2 - 165X^6Y^4Z - 54X^6Y^2Z^2 \\ -54X^6Y^6 - 189X^5Y^5 - 216X^5Y^3Z - 36X^5YZ^2 \\ -222X^4Y^4 - 117X^4Y^2Z - 9X^4Z^2 - 21X^3YZ - 89X^3Y^3 - 2 \end{array} \right. \\ + \\ \left\{ \begin{array}{l} 9X^8Y^4Z^2 + 45X^7Y^5Z + 36X^7Y^3Z^2 + 165X^6Y^4Z + 54X^6Y^2Z^2 \\ +54X^6Y^6 + 189X^5Y^5 + 216X^5Y^3Z + 36X^5YZ^2 \\ +222X^4Y^4 + 117X^4Y^2Z + 9X^4Z^2 + 21X^3YZ + 89X^3Y^3 \end{array} \right. \end{array} \right.$$

Therefore $|J| = -2$. The proof is complete. ∎

2 Retroduction

In order to get to the heart of the counter example faster, no customary introduction was provided. However we include some introductory comments in this section. This writeup is meant for a broader appeal, and is unlikely to be of any interest to the specialists. Perhaps, the most comprehensive overview of Jacobian conjecture is available in [BCW82]. First, we give the statement of the conjecture.

Conjecture 2.1 (Jocobian Conjecture)**.** Let $\mathbb{F}$ denote $\mathbb{R}$ the field of real numbers or $\mathbb{C}$ the field of complex numbers. Let

$$F_i := F_i\left(X_1, X_2, \ldots, X_n\right) \in \mathbb{F}\left[X_1, X_2, \ldots, X_n\right] \qquad i = 1, 2, \ldots, n$$

be n polynomials. Let

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial X_1} & \frac{\partial F_2}{\partial X_1} & \frac{\partial F_3}{\partial X_1} & \cdots & \frac{\partial F_n}{\partial X_1} \\ \frac{\partial F_1}{\partial X_2} & \frac{\partial F_2}{\partial X_2} & \frac{\partial F_3}{\partial X_2} & \cdots & \frac{\partial F_n}{\partial X_2} \\ \frac{\partial F_1}{\partial X_3} & \frac{\partial F_2}{\partial X_3} & \frac{\partial F_3}{\partial X_3} & \cdots & \frac{\partial F_n}{\partial X_3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_1}{\partial X_n} & \frac{\partial F_2}{\partial X_n} & \frac{\partial F_3}{\partial X_n} & \cdots & \frac{\partial F_n}{\partial X_n} \end{pmatrix}$$

be the Jacobian matrix, and $|J| \in \mathbb{F}[X_1, X_2, \ldots, X_n]$ denote the Jacobian determinants.

$$\begin{cases} \text{With} \quad \mathbf{F} = (F_1, F_2, \cdots, F_n) & \\ \mathbf{F} \text{ defines a map} & \text{by} \quad \mathfrak{F}(x) = (F_1(x), F_2(x), F_3(x), \cdots, F_n(x)) \\ \mathbb{F}^n \xrightarrow{\mathfrak{F}} \mathbb{F}^n & \end{cases}$$

The conjecture states that the following conditions are equivalent:

1. The Jacobian determinant $|J| \in \mathbb{F}^\star$ is a unit (*equivalently J is invertible*)

2. The map $\mathbf{F}$ has a polynomial inverse. This means that there are polynomials $G_i \in \mathbb{F}[X_1, X_2, \ldots, X_n]$ for $i = 1, 2, \ldots, n$, such that

$$\begin{cases} (F_1(\mathbf{G}), F_2(\mathbf{G}), F_3(\mathbf{G}), \cdots, F_n(\mathbf{G})) = (X_1, X_2, X_3, \ldots, X_n) \\ (G_1(\mathbf{F}), G_2(\mathbf{G}), G_3(\mathbf{F}), \cdots, G_n(\mathbf{F})) = (X_1, X_2, X_3, \ldots, X_n) \end{cases}$$

   In particular, the map $\mathfrak{F}$ is bijective.

■

We briefly, make a few remarks.

1. The example (1) and the theorem 1.1 means that conjecture (2.1) fails for $n = 3$. The conjecture is still open for $n = 2$.

2. Note that the validity of the 2^{nd}-condition implies the validity of the 1^{st}-condition, by chain rule.

3. It will not be far from the truth that there is essentially no results (or a class affirmative examples) on the conjecture (2.1). However, many equivalent formulations of the conjecture is available in the literature. They are fairly comprehensively summarized in [BCW82].

There is an easy consequence of Inverse function theorem.

Proposition 2.2. Assume $\mathbb{F} = \mathbb{R}$ and $F_i \in \mathbb{R}[X_1, X_2, \ldots, X_n]$ for $i = 1, 2, \ldots, n$. Assume the Jacobian determinant $|J|(\mathfrak{a}) \neq 0 \ \forall \mathfrak{a} \in \mathbb{R}^n$. Fix $\mathfrak{a}_0 \in \mathbb{R}^n$. Then there are open subsets $V, W \subseteq \mathbb{R}^n$, with $\mathfrak{a}_0 \in V$ and $\mathfrak{F}(\mathfrak{a}_0) \in W$ and the restriction

$$V \xrightarrow[\sim]{\mathfrak{F}_{|V}} W \quad \text{is a diffeomorphism.}$$

Proof. Follows from Inverse function theorem. ■

Under the hypothesis of the conjecture (2.1), there are some formal power series inverses due to R. V. Gurujar and S. S. Abhyankar, as follows.

Theorem 2.3. Let $\mathbb{F}$ denote $\mathbb{R}$ the field of real numbers or $\mathbb{C}$ the field of complex numbers. Let

$$F_i := F_i\left(X_1, X_2, \ldots, X_n\right) \in \mathbb{F}\left[X_1, X_2, \ldots, X_n\right] \qquad i = 1, 2, \ldots, n$$

be n polynomials, such that the Jacobian determianant $|J| \in \mathbb{F}^{\star}$ is a unit. Assume (*which can be achieved*)

$$F_i = X_i + \text{terms of } \deg \geq 2$$

Write

$$G_i = \sum_{(r_1, r_2, \ldots, r_n) \in \mathbb{N}^n} \frac{\frac{\partial^{p_1}}{\partial {X_1}^{r_1}} \frac{\partial^{r_2}}{\partial {X_2}^{r_2}} \cdots \frac{\partial^{r_n}}{\partial {X_n}^{r_n}} \left(X_i \left|J\right| (X_1 - F_1)^{r_1} (X_2 - F_2)^{r_2} \cdots (X_n - F_n)^{r_n}\right)}{r_1! r_2! \cdots r_n!}$$

Then $\mathbf{G} = (G_1, \ldots, G_n)$ is the inverse of $\mathbf{F} = (F_1, \ldots, F_n)$.

Proof. See [BCW82]. ■

References

[A26] Levent Alpöge, announcement, July 19, 2026, https://x.com/_alpoge_/status/2079028340955197566

[BCW82] Bass, Hyman; Connell, Edwin H.; Wright, David The Jacobian conjecture: reduction of degree and formal expansion of the inverse Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 2, 287–330.