Matrices: §2.2 Properties of Matrices

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Goals

We will discuss the properties of matrices with respect to addition, scalar multiplications and matrix multiplication and others. Among what we will see

1. Matrix multiplication do not commute. That means, not always $AB = BA$.
2. We will define **transpose** A^T of a matrix A and discuss its properties.

Algebra of Matrices

Let A, B, C be $m \times n$ matrices and c, d be scalars. Then

$$\begin{array}{ll} A+B=B+A & \textit{Commutativity of addition} \\ A+(B+C)=(A+B)+C & \textit{Associativity of addition} \\ (cd)A=c(dA) & \textit{Associativity of scalar multiplicatio} \\ c(A+B)=cA+cB & \textit{a Distributive property} \\ (c+d)A=cA+dA & \textit{a Distributive property} \end{array}$$

These seem obvious, expected and are easy to prove.

Zero

The $m \times n$ matrix with all entries zero is denoted by O_{mn}. For a matrix A of size $m \times n$ and a scalar c, we have

- $A + O_{mn} = A$ (This property is stated as: *O_{mn} is the additive identity in the set of all $m \times n$ matrices.*)
- $A + (-A) = O_{mn}$. (This property is stated as: *$-A$ is the additive inverse of A.*)
- $cA = O_{mn} \quad \Longrightarrow \quad c = 0 \quad \textit{or} \quad A = O_{mn}.$

Remark. So far, it appears that matrices behave like real numbers.

Properties of Matrix Multiplication

Let A, B, C be matrices and c is a constant. Assume all the matrix products below are defined. Then

$$\begin{array}{ll} A(BC) = (AB)C & \textit{Associativity Matrix Product} \\ A(B+C) = AB + AC & \textit{Distributive Property} \\ (A+B)C = AC + BC & \textit{Distributive Property} \\ c(AB) = (cA)B = A(cB) & \end{array}$$

Proofs would be routine checking (first one may be tedious). We skip the proofs.

Definition

For a positive integer, I_n would denote the square matrix of order n whose main diagonal (left to right) entries are 1 and rest of the entries are zero. So,

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of the Identity Matrix

Let A be a $m \times n$ matrix. Then

- $AI_n = A$
- $I_mA = A$
- If A is a square matrix of size $n \times n$, then

$$AI_n = I_nA = A.$$

- I_n is called the Identity matrix of order n. This is why, we say that I_n is the multiplicative identity for the set of all square matrices of order n.

Proof.

Proof. We will prove for 3×3 matrices A. Write

$$A = \begin{bmatrix} a & b & c \\ u & v & w \\ x & y & z \end{bmatrix} \qquad So,$$

$$AI_3 = \begin{bmatrix} a & b & c \\ u & v & w \\ x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ u & v & w \\ x & y & z \end{bmatrix} = A$$

Similarly, $I_3A = A$. The proof is complete. ■

Use of Matrix Algebra to solve systems of linear equation

Now that we are familiar with some Algebra of Matrices, we use it to give a proof of the following, stated before:

$0 - 1 - \infty$ **Theorem.**

For a system of linear equations (*with m equations in n variables*), precisely one of the following is true:

- The system has no solution.
- The system has exactly one solution.
- The system has an infinite number of solutions.

The Proof.

We write the system in the matrix form $A\mathbf{x} = \mathbf{b}$, where A is the coefficient matrix (with size $m \times n$),

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix}. \quad \begin{cases} \mathbf{x} \textit{ is the matrix of variables} \\ \mathbf{b} \textit{ is the constant matrix} \end{cases}$$

Continued

Suppose first two possibilities fail for the system $\mathbf{Ax} = \mathbf{b}$. That means, the system has at least two (distinct) solutions, namely $\mathbf{x_1}, \mathbf{x_2}$ with $\mathbf{x_1} \neq \mathbf{x_2}$. So,

$$A\mathbf{x_1} = \mathbf{b} \quad \textit{and} \quad A\mathbf{x_2} = \mathbf{b}.$$

With $\mathbf{y} = \mathbf{x_1} - \mathbf{x_2} \neq 0$ we have

$$A\mathbf{y} = A(\mathbf{x_1} - \mathbf{x_2}) = A\mathbf{x_1} - A\mathbf{x_2} = \mathbf{b} - \mathbf{b} = 0.$$

Now, for any scalar c, we have

$$A(\mathbf{x_1} + c\mathbf{y}) = A\mathbf{x_1} + cA\mathbf{y} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

So, given any real number c we have exhibited $\mathbf{x_1} + c\mathbf{y}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Therefore, the system has infinitely many solutions. The proof is complete. ∎

Definition of Transpose of a Matrix

Definition. Given a $m \times n$ matrix A, the transpose of A, denoted by A^T, is formed by writing the columns of A as rows (equivalently, writing the rows as columns). So, transpose A^T of the $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{13} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

Transpose of a Matrix: Continued

is the $n \times m$ matrix

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{pmatrix} \quad an\ n \times m\ matrix$$

Properties of Transpose

Let A, B be matrices and c be a scalar. Then,

$$\begin{array}{ll} (A^T)^T = A & \textit{Double transpose of A is itself} \\ (A+B)^T = A^T + B^T & \text{when } A+B \text{ is defined} \\ (cA)^T = cA^T & \textit{transpose of scalar multiplication} \\ (AB)^T = B^T A^T & \text{when } AB \text{ is defined} \end{array}$$

Again, to prove we check entry-wise equalities.
Caution: Note $(AB)^T$ is not $A^T B^T$.

Let me draw your attention, how algebra of matrices differ from that of the algebra of real numbers:

- Matrix product is not commutative. That means $AB \neq BA$, for some matrices A, B. See the Example below.
- Cancellation property fails. That means there are matrices A, B, C, with $C \neq \mathbf{0}$, such that

$$AC = BC \quad but \quad A \neq B.$$

See Example below.

Example of noncommutativity $AB \neq BA$:

We have

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Right hand sides of these two equations are not equal. So, commitativity fails for these two matrices.

Example: $AC = BC$ but $A \neq B$:

We have

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

So, cancellation property fails for matrix product.

Solve for Matrix X

Solve for the matrix X when

$$A = \begin{bmatrix} -1 & 1 \\ 3 & -1 \\ 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 11 & -1 \end{bmatrix}$$

- (1) $5X + 3A = 2B$
- (2) $2B + 4A = 5X$
- (3) $X + 2A - 2B = \mathbf{0}$

Solution

- For (1)

$$5X + 3A = 2B \Longrightarrow \quad X = \frac{2}{5}B - \frac{3}{5}A$$

$$= .4\begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 11 & -1 \end{bmatrix} - .6\begin{bmatrix} -1 & 1 \\ 3 & -1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.4 & .6 \\ -.2 & .2 \\ 3.2 & .2 \end{bmatrix}$$

Solution: Continued

- For (2), $X = \frac{2}{5}B + \frac{4}{5}A =$

$$.4\begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 11 & -1 \end{bmatrix} + .8\begin{bmatrix} -1 & 1 \\ 3 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & -1.2 \\ 6 & -1.2 \end{bmatrix}$$

- For (3) $X = -2A + 2B =$

$$-2\begin{bmatrix} -1 & 1 \\ 3 & -1 \\ 2 & -1 \end{bmatrix} + 2\begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 11 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 0 \\ 18 & 0 \end{bmatrix}$$

Example

Let

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 4 & -1 & 19 \\ 11 & -1 & -19 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 3 & -1 \\ 11 & 2 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Demonstrate $AC = BC$ but $A \neq B$.

Solution:

We have

$$AC = \begin{bmatrix} 2 & 3 & 7 \\ 4 & -1 & 19 \\ 11 & -1 & -19 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 2 \\ 8 & -4 & 4 \\ 22 & -11 & -11 \end{bmatrix},$$

$$BC = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 3 & -1 \\ 11 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 2 \\ 8 & -4 & 4 \\ 22 & -11 & -11 \end{bmatrix}$$

So, $AC = BC$. ■

Prelude

Given a polynomial $f(x)$, we are use to the idea of evaluating $f(2), f(3)$ or $f(a)$ for any real number a.
Likewise, we evaluate $f(A)$ for any square matrix A.

Example S1

Let $f(x) = x^2 - 2x + 2$

$$A = \begin{bmatrix} 4 & -2 & 0 \\ 8 & -4 & 4 \\ 22 & 0 & 0 \end{bmatrix}$$

Compute $f(A)$.

Solution: Since A is a square matrix of order 3, read $f(x)$ as:

$$f(x) = x^2 - 2x + 2I_3$$

Solution: Continued

We have

$$A^2 = \begin{bmatrix} 4 & -2 & 0 \\ 8 & -4 & 4 \\ 22 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 & 0 \\ 8 & -4 & 4 \\ 22 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -8 \\ 88 & 0 & -16 \\ 88 & -44 & 0 \end{bmatrix}$$

Solution: Continued

Let $f(A) = A^2 - 2A + 2I_3 =$

$$\begin{bmatrix} 0 & 0 & -8 \\ 88 & 0 & -16 \\ 88 & -44 & 0 \end{bmatrix} - 2 \begin{bmatrix} 4 & -2 & 0 \\ 8 & -4 & 4 \\ 22 & 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 4 & -8 \\ 72 & 10 & -24 \\ 44 & -44 & 2 \end{bmatrix}$$

Example S2

Let $f(x) = x^3 - 2x^2 + x + 1$

$$A = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Compute $f(A)$.

Solution: Since A is a square matrix of order 4, read $f(x)$ as:

$$f(x) = x^3 - 2x^2 + x + I_4$$

Solution: Continued

$$A^2 = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -4 & 0 & -1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution: Continued

$$A^3 = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -4 & 0 & -1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -6 & -3 & 2 \\ 0 & 1 & 3 & -6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution: Continued

$$f(x) = x^3 - 2x^2 + x + I_4 =$$

$$\begin{pmatrix} 1 & -6 & -3 & 2 \\ 0 & 1 & 3 & -6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & -4 & 0 & -1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$