Chapter 2 First Order ODE § 2.5 Existence and Uniqueness of Solutions

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First Order DE

- Recall the general form of the 1^{st} Order ODEs (FODE):

$$\frac{dy}{dt} = f(t, y) \tag{1}$$

- An ODE, with an initial value condition $y(t_0) = y_0$ is called an Initial Value Problem (IVP).
  So, a 1^{st} Order IVP looks like:

$$\frac{dy}{dt} = f(t, y) \qquad y(t_0) = y_0 \tag{2}$$

Purpose

In this section, we consider the following two Question:

- **Question of Existence:** Given a 1^{st} Oder IVP, as above (2), whether the IVP (2) has a solution $y = \varphi(t)$?
- **Question of Uniqueness:** In case the IVP (2) has a solution, is the solution unique?

In the case of 1^{st} order IVP, method of solutions (using integrating factors) show solution usually exists and is unique. But a definitive answer to these questions is much more restrictive. We state one such theorem, in the next frame.

An Existence and Uniqueness Theorem

Theorem 2.5.1: Consider the 1^{st}-order Linear IVP

$$\begin{cases} y' + p(t)y & = g(t) \\ y(t_0) & = y_0 \end{cases} \tag{3}$$

Assume $p(t), g(t)$ are continuous on an interval $I : \alpha < t < \beta$ and t_0 is in I. Then

- The IVP (3) has a solution $y = \varphi(t)$.
- The domain of $y = \varphi(t)$ is I.
- The solution $y = \varphi(t)$ is unique, on I.

Continued

The proof of the existence of a solution would be a repetition of the steps followed in § 2.1 to solve Linear ODEs.

The Case of Non-Linear ODEs

Now consider a nonlinear First Order IVP

$$\begin{cases} \frac{dy}{dt} & = f(t, y) \\ y(t_0) & = y_0 \end{cases}$$

- In general, existence of a solution $y = \varphi(t)$ for such an IVP is not guaranteed, without further conditions on $f(y, t)$.
- Even when a solution exists, it is not guaranteed that the solution $y = \varphi(t)$ would be unique. We would not state any other existence or uniqueness theorems.

- However, we saw that separable ODEs (§ 2.2) have solutions, whenever the respective integrals exists. Similarly, we saw Homogeneous and Bernoulli's ODEs (§ 2.4) have solutions (*under some restrictions that we ignored to state*).
- Likewise, we would see in future sections, that some other forms of ODEs have solutions.
- Again, things can be deceptive, as shown in the next example.

A Deceptive Example

Consider the IVP

$$\frac{dy}{dt} = -\frac{t}{y} \qquad y(0) = 0$$

This IVP seem to have a solution, while it does not.

Clirification: The ODE is separable. We have

$$\int y dy = -\int t dt + c \Longrightarrow \frac{y^2}{2} = -\frac{t^2}{2} + c$$

Now, $y(0) = 0$ implies $c = 0$

Continued

So, the solution to the ODE, in the implicit form is

$$\frac{y^2}{2} = -\frac{t^2}{2} \Longrightarrow y^2 = -t^2$$

So, in the explicit form, the solution is

$$y = \pm\sqrt{-t^2} \quad \text{which is not a real valued function.}$$

So, the IVP does not have a (real) solution.

Nature of Problems

We would use Theorem 2.5.1 to determine the interval, on which a Linear IVP has a solution.

Example 1

Consider the initial value problem (IVP)

$$\begin{cases} (t+1)(t-1)(t-2)\frac{dy}{dt} + e^{t^2}y & = \sin t^2 \\ y(3) & = 1 \end{cases}$$

Use Theorem 2.5.1 to determine the interval in which this IVP has (Do not try to solve).

- Write the equation in the standard form (3):

$$\begin{cases} \frac{dy}{dt} + \frac{e^{t^2}}{(t+1)(t-1)(t-2)}y & = \frac{\sin t^2}{(t+1)(t-1)(t-2)} \\ y(3) & = 1 \end{cases}$$

Continued

- $p(t) = \frac{e^{t^2}}{(t+1)(t-1)(t-2)}$ is not defined at $t = -1, 1, 2$.
  Likewise $g(t) = \frac{\sin t^2}{(t+1)(t-1)(t-2)}$ is not defined at $t = -1, 1, 2$.
  Split the number line as:
  $(-\infty, -1), (-1, 1), (1, 2), (2, \infty)$.
- Both $p(t)$, $g(t)$ are continuous on the intervals $(-\infty, -1), (-1, 1), (1, 2), (2, \infty)$.
- The initial t-value $t = 3$ is in $(2, \infty)$
- By theorem 2.5.1 the IVP has a unique solution on the interval $(2, \infty)$.

Example 2

Consider the initial value problem (IVP)

$$\begin{cases} \cos t \frac{dy}{dt} + y & = 1 + t^2 \\ y(\pi) & = 0 \end{cases}$$

Use Theorem 2.5.1 to determine the interval in which this IVP has (Do not try to solve).

- Write the equation in the standard form (3):

$$\begin{cases} \frac{dy}{dt} + \frac{1}{\cos t} y & = \frac{1+t^2}{\cos t} \\ y(\pi) & = 0 \end{cases}$$

Continued

- $p(t) = \frac{1}{\cos t}$ and $g(t) = \frac{1+t^2}{\cos t}$ are not defined at $t = \cdots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$.
- Accordingly, split the number line as:

$$\cdots, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \cdots$$

- Both $p(t)$ and$g(t)$ are continuous on these intervals.
- The initial t-value $t = \pi$ is in$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.
- By theorem 2.5.1 the IVP has a unique solution on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Nonlinear DE

- Nonlinear ODEs would not behave as nicely as in theorem 2.5.1.
- Uniqueness is not guaranteed.
- Solutions, if exist, may come out in an implicit form.
- Some ODEs may not have an analytic solution. In such cases, numerical solutions would be an option.