Chapter 4
Vector Spaces
§4.6 Rank of a Matrix II

Satya Mandal

U. Kansas
Arrowtic K-Theory
Fall 2025

Goals

Suppose A is a $m \times n$ matrix

- The row space and column space of A will be defined, as a vector space.
- $rank(A)$ will be re-defined. *This will agree with $rank(A)$ defined in the the chapter on determinants.*
- Then Null space $N(A)$ of A will be defined, as a vector space.
- It will be proved $rank(A) + dim(N(A)) = n$.

Row and Column vectors

Suppose A is a $m \times n$ matrix.

- The $n-$tuples corrospoding to the rows of A are called row vectors of A. So, the row vectors of A are in $\mathbb{R}^n$.
- The $m-$tuples corrospeding to the columns of A are called columns vectors of A. So, the column vectors of A are in $\mathbb{R}^m$.

Row and Column vectors

Suppose

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \textit{Then} \quad \begin{array}{c} \textit{row vectors of } A \\ (a_{11}, \quad a_{12}, \quad \cdots, \quad a_{1n}) \\ (a_{21}, \quad a_{22}, \quad \cdots, \quad a_{2n}) \\ \cdots \\ (a_{m1}, \quad a_{12}, \quad \cdots, \quad a_{mn}) \end{array}$$

The column vectors of A are

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{pmatrix}, \cdots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{pmatrix}$$

Row and Column Spaces

Suppose A is a $m \times n$ matrix, as above.

- The row space of A is defined to be the subspace of $\mathbb{R}^n$ spanned by row vectors of A.
  (*In this case, think of vectors in* $\mathbb{R}^n$, *as rows*.)
- The column space of A is defined to be the subspace of $\mathbb{R}^n$ spanned by column vectors of A.
  (*In this case, think of vectors in* $\mathbb{R}^n$, *as columns*.)

Continued

So,

- $row - space(A) =$

$$span \begin{pmatrix} (a_{11}, & a_{12}, & \cdots, & a_{1n}) \\ (a_{21}, & a_{22}, & \cdots, & a_{2n}) \\ & & \cdots & \\ (a_{m1}, & a_{12}, & \cdots, & a_{mn}) \end{pmatrix} \subseteq \mathbb{R}^n.$$

- $column - space(A) =$

$$span \left(\begin{pmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{pmatrix}, \cdots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{pmatrix} \right) \subseteq \mathbb{R}^m.$$

Some Properties: Theorem 4.6.1

Theorem 4.6.1 Suppose A, B are two $m \times n$ matrices.

- If A and B are row equivalent then, $rowSpace(A) = rowSpace(B)$.
- If A and B are column equivalent then, $columnSpace(A) = columnSpace(B)$.

Remark. The properties of row-spaces and column-spaces would be analogous.

Continued

Proof. To prove the first statement, assume A and B are row equivalent. We can write $B = E_k E_{k-1} \cdots E_2 E_1 A$, where $E_1, \ldots, E_k$ are elementary row matrices. Inductively, we prove

$$rowspace(A) = rowspace(E_1 A) = rowspace(E_2 E_1 A) = \cdots$$

$$= rowspace(E_{k-1} \cdots E_2 E_1 A) = rowspace(B).$$

So, it is enough to prove $rowspace(A) = rowspace(EA)$, where E is an elementary row matrix.

Continued

- If E switches two rows of I_m, EA switches two rows of A. Hence $rowspace(A) = rowspace(EA)$.
- If E is obtained by multiplying a row of I_m by $c \neq 0$, EA is obtained from multiplying the same row by c. Hence $rowspace(EA) \subseteq rowspace(A)$.
  Now, A is obtained, from EA by multiplying the same row of EA by $\frac{1}{c}$.
  So, $rowspace(A) \subseteq rowspace(EA)$, and $rowspace(EA) = rowspace(A)$.

Continued

- If E is obtained by adding c times a row of I_m, then EA is obtained, from A, by adding c times a row of A, to another. So, all rows of EA are either same or a linear combination of rows alf A. So, $rowspace(EA) \subseteq rowspace(A)$.
  Again, process is reversible. So, $rowspace(A) \subseteq rowspace(EA)$. Therefore, $rowspace(EA) = rowspace(A)$.

This completes the proof of the statement on row spaces.

Continued

Now, suppose B is column equivalent to A.
Then B^T is row equivalent to A^T. We have

$$Columnspace(A) = rowspace(A^T) = rowspace(B^T)$$

$$= Columnspace(B)$$

This completes the proof of the statement on column spaces.
■

Basis of Row Space: Theorem 4.6.2

(Basis of Row Space) Suppose A, B are two $m \times n$ matrices. If B is in row-echelon form then the non-zero rows of B forms a basis of $row - space(A)$ (which is same as $row - space(B)$).

Proof. Note that the nonzero rows of a matrix in Echelon form are linearly independent. Now, the theorem follows from Theorem 4.6.1. ∎

Theorem 4.6.3

Let A be $m \times n$ matrix. Then

$$\dim(rowSpace(A)) = \dim(columnSpace(A)).$$

Proof. Write

$$A = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \cdots \\ \mathbf{r}_m \end{pmatrix} = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{pmatrix}$$

where $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ are the rows of A and $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ are the columns of A

Continued

Assume $\dim(rowspace(A)) = k$.
Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be a basis of $rowspace(A)$ (*and think of* $\mathbf{u}_i$ *as row matrices*). We can write

$$\left\{ \begin{array}{c} \mathbf{r}_1 = \alpha_{11}\mathbf{u}_1 + \alpha_{12}\mathbf{u}_2 + \cdots + \alpha_{1k}\mathbf{u}_k \\ \mathbf{r}_2 = \alpha_{21}\mathbf{u}_1 + \alpha_{22}\mathbf{u}_2 + \cdots + \alpha_{2k}\mathbf{u}_k \\ \cdots \\ \mathbf{r}_m = \alpha_{m1}\mathbf{u}_1 + \alpha_{m2}\mathbf{u}_2 + \cdots + \alpha_{mk}\mathbf{u}_k \end{array} \right. \qquad \text{where} \quad \alpha_{ij} \in \mathbb{R}.$$

Continued

We rewrite it in matrix form:

$$A = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \cdots \\ \mathbf{r}_m \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1k} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mk} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \cdots \\ \mathbf{u}_k \end{pmatrix}$$

Let me caution you that, on right hand side, we are multiplying a matrix with real number entries , with a matrix whose entries are row vectors. However, it works. You can accept or trust it or try to convince yourself.

Continued

Apply Transpose operation to the above equation:

$$A^T = \begin{pmatrix} \mathbf{u}_1^T & \mathbf{u}_2^T & \cdots & \mathbf{u}_k^T \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{m1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{1k} & \alpha_{2k} & \cdots & \alpha_{mk} \end{pmatrix} \quad (1)$$

We denote the rows of the matrix (α_{ij}) by $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k$. So,

$$\begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{m1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{1k} & \alpha_{2k} & \cdots & \alpha_{mk} \end{pmatrix} = \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \cdots \\ \mathbf{w}_k \end{pmatrix}$$

Continued

Note, $\begin{pmatrix} \mathbf{u}_1^T & \mathbf{u}_2^T & \cdots & \mathbf{u}_k^T \end{pmatrix}$ is an $n \times k$ matrix. So, we write

$$\begin{pmatrix} \mathbf{u}_1^T & \mathbf{u}_2^T & \cdots & \mathbf{u}_k^T \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{21} & \cdots & \beta_{1k} \\ \beta_{12} & \beta_{22} & \cdots & \beta_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nk} \end{pmatrix} \text{ where } \beta_{ij} \in \mathbb{R}.$$

Continued

So, Equation 1 can be written as

$$\begin{pmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \cdots \\ \mathbf{c}_n^T \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1k} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nk} \end{pmatrix} \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \cdots \\ \mathbf{w}_k \end{pmatrix}$$

$$\text{It follows } \left\{ \begin{array}{l} \mathbf{c}_1^T = \beta_{11}\mathbf{w}_1 + \beta_{12}\mathbf{w}_2 + \cdots + \beta_{1k}\mathbf{w}_k \\ \mathbf{c}_2^T = \beta_{21}\mathbf{w}_1 + \beta_{22}\mathbf{w}_2 + \cdots + \beta_{2k}\mathbf{w}_k \\ \cdots \\ \mathbf{c}_n^T = \beta_{n1}\mathbf{w}_1 + \beta_{n2}\mathbf{w}_2 + \cdots + \beta_{nk}\mathbf{w}_k \end{array} \right.$$

Continued

$$\text{Hence}\ \begin{cases} \mathbf{c}_1 = \beta_{11}\mathbf{w}_1^T + \beta_{12}\mathbf{w}_2^T + \cdots + \beta_{1k}\mathbf{w}_k^T \\ \mathbf{c}_2 = \beta_{21}\mathbf{w}_1^T + \beta_{22}\mathbf{w}_2^T + \cdots + \beta_{2k}\mathbf{w}_k^T \\ \cdots \\ \mathbf{c}_n = \beta_{n1}\mathbf{w}_1^T + \beta_{n2}\mathbf{w}_2^T + \cdots + \beta_{nk}\mathbf{w}_k^T \end{cases}$$

So,

$$ColumnSpan(A) = span\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} \subseteq span\{\mathbf{w}_1^T, \mathbf{w}_2^T, \ldots, \mathbf{w}_k^T\}$$

Therefore, by Corollary 4.5.7

$$\dim(ColumnSpan(A)) \leq k = \dim(rowSpace(A))$$

Continued

Now,

$$\dim(rowspace(A)) = \dim(ColumnSpace(A^T)) \leq$$

$$\dim(rowSpace(A^T)) = \dim(ColumnSpace(A))$$

Hence

$$\dim(ColumnSpan(A)) = k = \dim(rowSpace(A))$$

The proof is complete. ∎

Rank of a Matrix:

Definition. Let A be $m \times n$ matrix. Then the rank of A (written as $rank(A)$) is defined to be the dimension of the row space of A (equivalently, dimension of the column space of A).

$$rank(A) := \dim(rowSpace(A))$$

By Theorem 4.6.3, we also have,

$$\begin{cases} rank(A) = \dim(ColumnSpace(A)) \\ rank(A^t) = rank(A) \end{cases}$$

Two Definitions of Rank

- For a matrix A, in chapter 3 (on Determinants), we defined

$$rank(A) = \max\{k : \exists \text{ a minor M of order } k, \ni \text{ M} \neq 0\}$$

- **Theorem.** These two definitions of $rank(A)$ coincide. You can compute $rank(A)$, in either way.

Null Space of a Matrix

Definitions Let A be an $m \times n$ matrix.
The Null Space $N(A)$ is defined to be the set of all the solutions of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$.

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

The null space $N(A)$ is also called the solution space of $A\mathbf{x} = \mathbf{0}$.

Theorem 4.6.4

Suppose A is am $m \times n$ matrix.
Then the null space $N(A)$ is a subspace of $\mathbb{R}^n$.
Proof. By defintion $N(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$.
Note that $\mathbf{0}$ is a solution of $A\mathbf{x} = \mathbf{0}$, $\mathbf{0} \in N(A)$
(called the trivial solution). So, $N(A)$ is nonempty.

Continued

Now, we only need to show that $N(A)$ is closed under addition and scalar multiplication. Suppose $\mathbf{x}, \mathbf{y} \in N(A)$ and c is a scalar. Then

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

So, $\mathbf{x} + \mathbf{y} \in N(A)$. So, $N(A)$ is closed under addition. Also,

$$A(c\mathbf{x}) = cA(\mathbf{x}) = c\mathbf{0} = \mathbf{0}$$

So, $c\mathbf{x} \in N(A)$. So, $N(A)$ is closed under scalar multiplication. Therefore $N(A)$ is a subspace of $\mathbb{R}^n$.
The proof is complete.

Definition: Nullity

Let A be an $m \times n$ matrix.
Then nullity of A (written, $Nullity(A)$) is defined to be the dimension of the null space of A. Notationally,

$$Nullity(A) = \dim(N(A)).$$

- The right side makes sense, because $N(A)$ is subspace.
- Now, given a matrix A, we have associated two numbers: (1) $rank(A)$ and (2) $Nullity(A)$.

Theorem 4.6.5

Let A be $m \times n$ matrix.

- Then

$$rank(A) + Nullity(A) = n. \quad \textit{The number of columns}$$

- This means, if rank of A is r, then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $n - r$.
- Note n is the number of variables in the system $A\mathbf{x} = \mathbf{0}$.

Proof.

Suppose A is row equivalent to B, which is in row Echelon form. We have seem (1) $rank(A) = rank(B)$, (2) $N(A) = N(B)$. So, $Nullity(B) = Nullity(A)$.
So, it is enough to prove the theorem, when A is in row Echelon form. In this case,

- $rank(A) = r$ is the number of non zero rows of A.
- Inspecting the system in Echelon form, it follows $\dim(N(A)) = n - r$. (*We skip the details.*)

■

Solutions of Linear Systems

Let $A\mathbf{x} = \mathbf{b}$ be a system of linear m equations, in n variables.

Remarks:

- Here A is a $m \times n$ matrix.
- For a homogeneous system $A\mathbf{x} = \mathbf{0}$, set of its solutions is the null space $N(A)$, which is a subspace of $\mathbb{R}^n$.
- If the system $A\mathbf{x} = \mathbf{b}$ is non-homogeneous (i.e. $\mathbf{b} \neq 0$), then the set of solutions is not a subspace of $\mathbb{R}^n$. This is because the zero $\mathbf{0}$ is not a solution of such a system.
- However, the solutions of a system $A\mathbf{x} = \mathbf{b}$ is related to the solutions space of he corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$, the null space $N(A)$.

Theorem 4.6.6: Solutions of Linear Systems

Let $A\mathbf{x} = \mathbf{b}$ be a system of linear m equations, in n variables. Fix a solution $\mathbf{x_p}$ of $A\mathbf{x} = \mathbf{b}$, to be called a "particular solution". Then any solution of $A\mathbf{x} = \mathbf{b}$ can be written as

$$\mathbf{x} = \mathbf{x_p} + \mathbf{x_h} \qquad\qquad \mathbf{x_h} \in N(A).$$

Proof. Suppose $A\mathbf{x} = \mathbf{b}$. Write $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p$. Then

$A\mathbf{x}_h = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}$. So, $\mathbf{x}_h \in N(A)$, and $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$.

This completes the proof. ∎

Theorem 4.6.7: Consistent System

A system $A\mathbf{x} = \mathbf{b}$ is consistent, if $\mathbf{b} \in ColumnSpace(A)$.
Proof. If the system has a solution $\mathbf{x}$, then $\mathbf{b}$ is linear combination of the columns: $\mathbf{b} = x_1(First - Column) + x_2(second - Column) + \cdot + x_n(First - Column)$. The proof is complete. ■

A Summary

Let A be a square marix of order n. The following conditions are equivalent:

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution.
3. $A\mathbf{x} = \mathbf{0}$ has only trivial solution.
4. A is row equivalent to I_n.
5. $|A| \neq 0$.
6. $rank(A) = n$.
7. The n row vectors of A are linearly independent.
8. The n column vectors of A are linearly independent.

Continued

- This (the last frame) is an update of a list given in §3.3.
- Last three conditions were added to the old list.
- From definition of rank, it follows

$$(6) \iff (7) \iff (8).$$

Also,

$$(4) \iff (7).$$

This completes the proof. ∎

Basis of row spaces

Suppose A is an $m \times n$ matrix.

- By Theorem 4.6.1, if B is in row-echelon form (or in reduced row-echelon form) and is row equivalent to A then the non-zero rows of B forms a basis of $row - space(A)$
  Remark. In fact, if B is like a row-echelon matrix, but without "leading 1", same works.
  We will call them essentially in row Echelon form.
  This sometimes help, avoiding decimals.

Example 4.6.1

$$\text{Let} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & -15 \\ 2 & 6 & -8 \end{bmatrix}. \quad \textit{Find basis of the row space of } A.$$

1. Find a basis of the row space of A.
2. Find the rank of A.

Solution.

Use "ref" in TI-84 ("ref")I.

$$B = \begin{bmatrix} 1 & \frac{7}{2} & -\frac{15}{2} \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A\ basis\ of\ the\ row\ pace: \quad S = \left\{ \left(1, \frac{7}{2}, -\frac{15}{2}\right), (0, 1, -7) \right\}$$

So, $rank(A) = \dim(rowSpace(A)) = 2$.

Example 4.6.2

Let A be as in Example 4.6.1. Now find a basis of the column space of A.

Solution. Column space of A is the row space of A^T (written as rows). We have

$$A^T = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & 6 \\ 3 & -15 & -8 \end{pmatrix}$$

I want to solve it by hand, without TI-84. TI-84 quickly brings in nonterminating decimal numbers, which is inconvenient.

Continued

Solution: Subtract 2 times first row from second and three times first row from third:

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & 2 \\ 0 & -21 & -14 \end{pmatrix}$$

Now subtract 7 times second row from third:

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{This is essentially in row Echelon form.}$$

Continued

So, a basis of *rowspace*(A^T) is

$$S = \{(1, 2, 2), (0, 3, 2)\}$$

So, a basis of the Column Space of A is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \right\}$$

Example 4.6.3

$$Let \quad A = \begin{pmatrix} 8 & 0 & 4 & 6 & 2 \\ 3 & 1 & 0 & -2 & -1 \\ 9 & 2 & 4 & 4 & 0 \\ 5 & 2 & 2 & -2 & -1 \\ 3 & 0 & 0 & -2 & -1 \end{pmatrix}$$

1. Find a basis of the row space of A
2. Find the rank of A.
3. Find the Nullity of A

Proof. We reduce A to an, essentially, row echelon form. Divide first row by two:

$$\begin{pmatrix} 4 & 0 & 2 & 3 & 1 \\ 3 & 1 & 0 & -2 & -1 \\ 9 & 2 & 4 & 4 & 0 \\ 5 & 2 & 2 & -2 & -1 \\ 3 & 0 & 0 & -2 & -1 \end{pmatrix}$$

Subtract 2 times 1^{st} row of A from 3^{rd} , the switch 1^{st} and 3^{rd} rows:

$$\begin{pmatrix} 4 & 0 & 2 & 3 & 1 \\ 3 & 1 & 0 & -2 & -1 \\ 1 & 2 & 0 & -2 & -2 \\ 5 & 2 & 2 & -2 & -1 \\ 3 & 0 & 0 & -2 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 0 & -2 & -2 \\ 3 & 1 & 0 & -2 & -1 \\ 4 & 0 & 2 & 3 & 1 \\ 5 & 2 & 2 & -2 & -1 \\ 3 & 0 & 0 & -2 & -1 \end{pmatrix}$$

(1) Subtract 3 times first row from second and fifth, and 4 times first row from third and 5 time first from fourth; (2) then subtract fifth row from second:

$$\begin{pmatrix} 1 & 2 & 0 & -2 & -2 \\ 0 & -5 & 0 & 4 & 5 \\ 0 & -8 & 2 & 11 & 9 \\ 0 & -8 & 2 & 8 & 9 \\ 0 & -6 & 0 & 4 & 5 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -8 & 2 & 11 & 9 \\ 0 & -8 & 2 & 8 & 9 \\ 0 & -6 & 0 & 4 & 5 \end{pmatrix}$$

Add 8 times 2^{nd} row to 3^{rd}, 4^{th} and 6 times 2^{nd} row to 5^{th}:

$$\begin{pmatrix} 1 & 2 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 11 & 9 \\ 0 & 0 & 2 & 8 & 9 \\ 0 & 0 & 0 & 4 & 5 \end{pmatrix}$$

Subtract 3^{rd} row from 4^{th}:

$$\begin{pmatrix} 1 & 2 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 11 & 9 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 4 & 5 \end{pmatrix}$$

Add $\frac{4}{3}$ times 4^{th} row to 5^{th}:

$$\begin{pmatrix} 1 & 2 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 11 & 9 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

The matrix is, essentially, in row Echelon form. So, its nonzero vectors from a basis of the $rowspace(A)$.

So, a basis of the row space of A is:

$$S = \left\{ \begin{array}{l} (1, 2, 0, -2, -2), \\ (0, 1, 0, 0, 0), \\ (0, 0, 2, 11, 9), \\ (0, 0, 0, -3, 0), \\ (0, 0, 0, 0, 5) \end{array} \right\}$$

So,

$$\begin{cases} rank(A) = \dim(row - space) = 5, \\ Nullity(A) = 5 - 5 = 0 \end{cases}$$

Example 4.6.3a

We modify the above problem, to create a new problem.

$$Let \quad A = \begin{pmatrix} 8 & 0 & 4 & 6 & 2 \\ 3 & 1 & 0 & -2 & -1 \\ 9 & 2 & 4 & 4 & 0 \end{pmatrix}$$

1. Find a basis of the row space of A
2. Find the rank of A.
3. Find the Nullity of A

Proof. We reduce A to an, essentially, row echelon form. Divide first row by two:

$$\begin{pmatrix} 4 & 0 & 2 & 3 & 1 \\ 3 & 1 & 0 & -2 & -1 \\ 9 & 2 & 4 & 4 & 0 \end{pmatrix}$$

Subtract 2^{nd} row from 1^{st} and subtract 3 times 2^{nd} row from from third:

$$\begin{pmatrix} 1 & -1 & 2 & 5 & 2 \\ 3 & 1 & 0 & -2 & -1 \\ 0 & -1 & 4 & 10 & 3 \end{pmatrix}$$

Switch 2^{nd} and 3^{rd} rows:

$$\begin{pmatrix} 1 & -1 & 2 & 5 & 2 \\ 0 & -1 & 4 & 10 & 3 \\ 3 & 1 & 0 & -2 & -1 \end{pmatrix}$$

Subtract 3 times 1^{st} from 3^{rd}:

$$\begin{pmatrix} 1 & -1 & 2 & 5 & 2 \\ 0 & -1 & 4 & 10 & 3 \\ 0 & 4 & -6 & -17 & -7 \end{pmatrix}$$

Add four times 2^{nd} to 3^{rd}:

$$\begin{pmatrix} 1 & -1 & 2 & 5 & 2 \\ 0 & -1 & 4 & 10 & 3 \\ 0 & 0 & 10 & -23 & 5 \end{pmatrix}$$

The matrix above is, essentially, in row Echelon form. So, its nonzero vectors from a basis of the $rowspace(A)$.
So, a basis of the row space of A is:

$$S = \left\{ \begin{array}{l} (1, -1, 2, 5, 2) \\ (0, -1, 4, 10, 3) \\ (0, 0, 10, -23, 5) \end{array} \right\}$$

So,

$$\left\{ \begin{array}{l} rank(A) = \dim(row - space) = 3, \\ Nullity(A) = 5 - 3 = 2 \end{array} \right.$$

Basis of subspaces

Suppose V is subspace of $\mathbb{R}^n$, spanned by a few given vectors. To find a basis of V do the following:

- Form a matrix A with these vectors, as rows.
- Then, row space of A is V.
- A basis of the row space would be a basis of V.

Example 4.6.4

Let $S = \{(3,2,2),(6,5,-1),(1,1,-1)\}$. Find a basis of $span(S)$, and $\dim(span(S))$.

Solution. Form the matrix A, with these rows.

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 6 & 5 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

Solution: We try to reduce the matrix, to a matrix essentially in Echelon form.

Continued

Switch first and third rows:

$$\begin{pmatrix} 1 & 1 & -1 \\ 6 & 5 & -1 \\ 3 & 2 & 2 \end{pmatrix}$$

Subtract 6 times 1^{st} row, from 2^{nd} and 3 times 1^{st} row, from 3^{rd}:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 5 \\ 0 & -1 & 5 \end{pmatrix}$$

Continued

Subtract 2^{nd} row from 3^{rd}:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix is essentially in row Echelon form. So,

$$\left\{ \begin{array}{l} Basis\ of\ \ span(S) = \{(1,1,-1),(0,-1,5)\} \\ \dim(span(S)) = 2 \end{array} \right.$$

Example 4.6.5

$$\text{Let} \qquad A = \begin{pmatrix} 7 & 14 & -21 \\ 3 & 6 & -9 \\ -4 & -8 & 12 \end{pmatrix}$$

- Give basis of the row space of A
- Compute $rank(A)$
- Compute $Nullity(A)$
- Give a basis of the null space $N(A)$.

Solution. First, reduce A to an, essentially, row echelon form.

Continued

Subtract 2 times 2^{nd} row of A from first:

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & 6 & -9 \\ -4 & -8 & 12 \end{pmatrix}$$

Subtract 3 times first row from second and, add 4 times first row to third:

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix is essentially in row echelon form.

Continued

- So, a Basis of the row space of A is $\{(1, 2, -3)\}$
- $rank(A) = 1$
- $Nullity(A) = 3 - 1 = 2$
- To compute $N(A)$ and a a basis of $N(A)$, we solve the homogeneous system for the above reduced matrix, which we do in the next frame.

Continued

The null space is the solution of the system, of one equation:

$$x + 2y - 3z = 0, \qquad \text{Or} \quad x = 3z - 2y$$

$$\text{With } y = t, \ z = s, x = 3s - 2t$$

$$\text{So,} \quad N(A) = \left\{ \begin{pmatrix} 3s - 2t \\ t \\ s \end{pmatrix} : t, s \in \mathbb{R} \right\}$$

Continued

Two comments:

- We already know, $nullity(A) = 2$. So, the basis of $N(A)$ would have two vectors.
- Intuitively, two bases will be obtained by taking $t = 1, s = 0$ and $t = 0, s = 1$.

A basis of $N(A)$ is:

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Example 4.6.6

Let $$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & -1 & 7 & 4 \\ 2 & 1 & 5 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

- Give basis of the row space of A
- Compute $rank(A)$
- Compute $Nullity(A)$
- Give a basis of the null space $N(A)$.

Continued

Solution. First, reduce A to an, essentially, row echelon form. Let me remind you that I avoided using TI-84, because, it brings in decimal numbers too quickly. As usual, subtract 4 times 1^{st} row from 2^{nd}, 2 times 1^{st} row from 3^{rd}, 1 times 1^{st} row from 4^{th}:

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -9 & -5 & 4 \\ 0 & -3 & -1 & 2 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

Continued

Switch 2^{nd} and 4^{th} row; them multiply the new 2^{nd} row by -1:

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -3 & -1 & 2 \\ 0 & -9 & -5 & 4 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -1 & 2 \\ 0 & -9 & -5 & 4 \end{pmatrix}$$

Add 3 times 2^{nd} row to 3^{rd} and 9 time 2^{nd} row to 4^{th}:

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

Continued

Subtract, 2 times 3^{rd} row from 4^{th}:

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix is essentially in row Echelon form.

Continued

- So, a Basis of the row space of A is $\{(1,2,3,0),(0,1,1,0),(0,0,2,2)\}$
- $rank(A) = 3$
- $Nullity(A) = 4 - 3 = 1$
- To compute $N(A)$ and a a basis of $N(A)$, we solve the homogeneous system for the above reduced matrix, which we do in the next frame.

Continued

$$\begin{pmatrix} x_1 & +2x_2 & +3x_3 & & =0 \\ 0 & x_2 & +x_3 & & =0 \\ 0 & 0 & 2x_3 & +2x_4 & =0 \end{pmatrix}$$

$$\text{With} \quad x_4 = t \quad \begin{cases} x_1 = -2t + 3t = t \\ x_2 = t \\ x_3 = -t \\ x_4 = t \end{cases}$$

Continued

The null space of A:

$$N(A) = \left\{ \begin{pmatrix} t \\ t \\ -t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

Since $nullity(A) = 1$, we are looking for one basis element. A basis of $N(A)$ is (obtained by taking $t = 1$):

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Example 4.6.7

$$\text{Let} \qquad A = \begin{pmatrix} 4 & -1 & 2 \\ 2 & 3 & -1 \\ 6 & 2 & 1 \end{pmatrix}$$

- Give basis of the row space of A
- Compute $rank(A)$
- Compute $Nullity(A)$
- Give a basis of the null space $N(A)$.

Continued

Solution. First, reduce A to an, essentially, row echelon form. Switch first and second row of A:

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & -1 & 2 \\ 6 & 2 & 1 \end{pmatrix}$$

Subtract, 2 times first row from second and 3 times first row from third:

$$\begin{pmatrix} 2 & 3 & -1 \\ 0 & -7 & 4 \\ 0 & -7 & 4 \end{pmatrix}$$

Continued

Subtract second row form third:

$$\begin{pmatrix} 2 & 3 & -1 \\ 0 & -7 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix is, essentially in row Echelon form.

Continued

- So, a Basis of the row space of A is $\{(2,3,-1),(0,-7,4)\}$
- $rank(A) = 2$
- $Nullity(A) = 3 - 2 = 1$
- To compute $N(A)$ and a a basis of A, we solve the reduced system above, which we do in the next frame.

Continued

Subtract second row form third:

$$\left\{\begin{array}{cccc} 2x & +3y & -z & =0 \\ & -7y & +4z & =0 \end{array}\right. \Longrightarrow \left\{\begin{array}{l} x=-\frac{3}{2}y+\frac{1}{2}z=-\frac{5}{14}z \\ y=\frac{4}{7}z \end{array}\right.$$

Taking $z=t$, we have

$$N(A)=\left\{\begin{pmatrix} -\frac{5}{14}t \\ \frac{4}{7}t \\ t \end{pmatrix} : t\in\mathbb{R}\right\}=\left\{\begin{pmatrix} -\frac{5}{14} \\ \frac{4}{7} \\ 1 \end{pmatrix} t : t\in\mathbb{R}\right\}$$

Continued

Since $nullity(A) = 1$, there is only one element in the basis of $N(A)$, which can be obtained by taking $t = 1$. A basis of $N(A)$ is:

$$\left\{ \begin{pmatrix} -\frac{5}{14} \\ \frac{4}{7} \\ 1 \end{pmatrix} \right\}$$

Alternately, we could take $t = 14$ and another basis of $N(A)$ would be

$$\left\{ \begin{pmatrix} -5 \\ 8 \\ 14 \end{pmatrix} \right\}$$