Chapter 3
Second Order ODE
§3.1 Introduction to Second Order ODE

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Second Order DE

- For many, the first encounter with second order ODE occurs, as one starts getting familiar with the concept of acceleration. Recall, acceleration is $\frac{d^2y}{dt^2}$ where y is the distance travelled. Second Order ODE (SODE) has wide range of applications, in the undergraduate courses in Physics and Engineering, for the same reason.
- SODE models are used in fluid dynamics, heat equations, wave motion, economics and so on.
- More importantly, a wide variety of SODEs can be solved by analytically.

Definition: SODEs

Definition. A second order ODE has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, y'\right) \tag{1}$$

where f is a function of $t, y, y' := \frac{dy}{dt}$.

Linear SODEs

- A SODE (1) is called a linear SODE (LSODE), if f is linear. That means, if DE (1) has the form:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) = g(t) - q(t)y - p(t)\frac{dy}{dt}$$

where $g(t), p(t), q(t)$ are functions of t.

Standard form of LSODEs

Such LSODEs are often written in the following forms:

- First,
$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \qquad (2)$$
where $p(t), q(t), g(t)$ are functions of t. Then also as:
$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = G(t) \qquad (3)$$
where $P(t), Q(t), R(t), G(t)$ are functions of t.
- The LSODE (3) can be reduced to (2), when $P(t) \neq 0$, and conversely.

Recall and Compare:

Recall the form of the first order linear ODE and compare:

$$\begin{cases} 1^{\text{st}} \text{ Order Linear}: & \frac{dy}{dt} + p(t)y = g(t) \\ 2^{\text{nd}} \text{ Order Linear}: & \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \end{cases}$$

Initial value problems in SODEs

- An initial value problem (IVP) in SODE consists of ODE (1), (2), or (3) together with initial value conditions:

$$y(t_0) = y_0, \quad y'(t_0) = y'_0.$$

So, one such form of a second order IVP is:

$$\begin{cases} \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \\ y(t_0) = y_0, \quad y'(t_0) = y'_0. \end{cases} \tag{4}$$

- As a rule of Thumb, two equations are needed to determine two unknowns. The general solutions of (1) involve two arbitrary constants c_1, c_2.
That is why (4) has two conditions.

Homogeneous LSODEs

- A LSODE is called homogeneous, if the right hand side, $g(t) = 0$ in (2), or if $G(t) = 0$ in (3).
- If $g(t) \neq 0$ [resp. $G(t) \neq 0$], the equations (2) [resp. (3)] would be called a nonhomogeneous linear equation.
- So, the a homogeneous LSODE can be written as

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0 \qquad (5)$$

- In fact (§ 3.5?), solutions of homogeneous LSODE (5) leads to solutions of nonhomogeneous LSODE (3).

Rest of this Chapter

- In next few sections, we solve homogeneous equations (5) with constant coefficients. So, they look like:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad a, b, c \in \mathbb{R}. \tag{6}$$

  However, we **comment** on ODEs (5), as is.

- In latter sections, we solve nonhomogeneous equations, whose left side is as in (6). So, they look like:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t) \tag{7}$$