Chapter 4
Higher Order ODE
§4.3 Nonhomogeneous Linear ODE

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Goals

In this section we discuss methods of solving nonhomogeneous Linear ODE, of higher order.
As in the case of 2^{nd}-order ODE, there are two methods:

- Method of Variation of Parameters, which we would state.
- Method of Undetermined Coefficients.
  We would only comment on it.

We would mostly consider nonhomogeneous Linear ODE, with constant coefficients.
Again, our goal is to provide an outline and flavor.

Linear ODE of Order n

Recall, a Nonhomogeneous Linear ODE of order n can be written as:

$$\mathcal{L}(y) = g(t) \quad \text{with} \quad g(t) \neq 0, \quad \text{where} \tag{1}$$

$$\begin{cases} \mathcal{L} := \frac{d^n}{dt^n} + p_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + p_1(t)\frac{d}{dt} + p_0(t) \\ \mathcal{L} := P_n(t)\frac{d^n}{dt^n} + P_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + P_1(t)\frac{d}{dt} + P_0(t) \end{cases} \tag{2}$$

We usually assume that $p_i(t)$, $P_i(t)$, $g(t)$ are continuous on an open interval I.

Definition

Definition A nonHomogeneous Linear ODE (1) is said to have constant coefficient, if $p_i(t), P_i(t)$ are constant functions. So, a linear Homogeneous ODE, of order n, with constant coefficients looks like

$$\mathcal{L}(y) = a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = g(t) \quad (3)$$

with $a_0, a_1, \cdots, a_n \in \mathbb{R}$, $a_n \neq 0$ and $g(t) \neq 0$.

The Corresponding Homogeneous ODE

Corresponding to the Homhomgeneous Linear ODE (1), there is homogeneous Linear ODE

$$\mathcal{L}(y) = 0 \tag{4}$$

We would see subsequently, that solutions of the Homhomgeneous Linear ODE (1) is derived from the corresponding homogeneous Linear ODE (4).

Role of the Homogeneous Part

The role of the corresponding homogeneous equation:

- Theorem 4.3.1 Let Y_1, Y_2 be two solutions of the nonhomogeneous Linear ODE (1)

$$\mathcal{L}(y) = g(t). \qquad \text{Then } Y_1 - Y_2 \text{ is a solution}$$

of the corresponding homogeneous ODE $\mathcal{L}(y) = 0$ (4).

- Proof. $\mathcal{L}(Y_1) = g(t), \ \ \mathcal{L}(Y_2) = g(t) \implies$

$$\mathcal{L}(Y_1 - Y_2) = \mathcal{L}(Y_1) - \mathcal{L}(Y_2) = g(t) - g(t) = 0.$$

The General Solution

Theorem 4.3.2

- Fix a (**particular**) solution Y of the nonhomogeneous Linear ODE (1) $\mathcal{L}(y) = g(t)$, of order n.
- Let $y = y_1, y = y_2, \ldots, y = y_n$ be a fundamental set of solutions of the homogeneous equation (4) $\mathcal{L}(y) = 0$.

Then the general solution of (1) is:

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots c_n y_n(t) + Y(t) \qquad (5)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.
Use the notation $y_c = \sum_{i=1}^{n} c_i y_i(t)$.

Method of Solutions

As mentioned above, we comment of two methods:

- Method of Variation of Parameters.
- Method of Undetermined Coefficients.

Theorem 4.3.3: Variation of Parameters

Analogous to the method of variation of parameters,
for 2^{nd}-Order Linear ODE,
for higher order we have:
Theorem 4.3.3: Consider former of the two forms of the nonhomogeneous Linear ODE (1), of order n. That means,

$$\begin{cases} \mathcal{L}(y) = g(t), \quad \text{with} \\ \mathcal{L} := \frac{d^n}{dt^n} + p_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + p_1(t)\frac{d}{dt} + p_0(t) \end{cases} \tag{6}$$

- Assume $p_i(t)$, $g(t)$ are continuous on an open interval I.
- Let $y = y_1, y = y_2, \ldots, y = y_n$ be a fundamental set of solutions of the homogeneous ODE $\mathcal{L}(y) = 0$.

Continued

Then: A particular solution of (6) is given by

$$Y = \sum_{i=1}^{n} y_i(t) \int \frac{\omega_i(t) g(t) dt}{W(t)} \qquad \text{where} \tag{7}$$

- $W(t) := W(y_1, y_2, \ldots, y_n)$ is Wronskian of $y_1, y_2, \ldots, y_n$.
- And, $\omega_i(t)$ denotes the cofactor of $y_i^{(n-1)}$ in the Wronskian matrix.

Continued

- So, by (5), the general solution of (6) is

$$y = y_c + Y = \sum_{i=1}^{n} c_i y_i + Y \tag{8}$$

Solving Problems

We worked out and assigned some problems on this topic, for 2^{nd}-order Linear Homogeneous ODE. For order $n \geq 3$, methods will be same, while it would be further laborious. For this reasons, we would not give any additional examples or exercises in this section.

Continued

Main points are:

- In this course, we only solve problems on nonhomgeneous Linear ODE, with constant coefficients (3) $\mathcal{L}(y) = g(t)$. Use Theorem 4.3.3 (Equation 7) to find a particular solution $y = Y$.
- For such a Linear ODE, constant coefficients (3), consider the homogeneous ODE $\mathcal{L}(y) = 0$. We elaborated methods to find a fundamental set of solutions $y = y_1, \ldots, y = y_n$, for $\mathcal{L}(y) = 0$. Now use Equation 5 to find a general solution.

Method Of Undetermined Coefficients

Method of Undetermined Coefficients, for higher order Linear ODE, with constant coefficients run similar to that of 2^{nd}-order Linear ODE, with constant coefficients. An interested reader can look at internet or any standard Textbook.