Chapter 5:
System of 1^{st}-Order Linear ODE
§5.4 The Theoretical Foundation

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System of 1^{st}-order Linear ODE

The goal of this section is to establish
the basic foundation of the system of 1^{st}-Order Linear ODE.
This theoretical foundation is fairly intuitive, and
is analogous to what we have seen in this course before.

System of 1^{st}-order Linear ODE

Recall (§5.1), a system 1^{st}-order linear ODE looks like:

$$\begin{cases} y_1' & = p_{11}(t)y_1 & +p_{12}(t)y_2 & +\cdots & +p_{1n}(t)y_n & +g_1(t) \\ y_2' & = p_{21}(t)y_1 & +p_{22}(t)y_2 & +\cdots & +p_{2n}(t)y_n & +g_2(t) \\ \cdots & & \cdots & & \cdots & \\ y_n' & = p_{n1}(t)y_1 & +p_{n2}(t)y_2 & +\cdots & +p_{nn}(t)y_n & +g_n(t) \end{cases} \tag{1}$$

This is a system of n Equations,
in n unknown variables $y_1, \ldots, y_n$.
The system (1) would be called Homogeneous,
if $g_1 = \cdots = g_n = 0$.

Continued

- Assume, $p_{ij}(t), g_j(t)$ are continuous on an interval $I : \alpha < t < \beta$.
- The system (1) can be written in the matrix form

$$\mathrm{y}' = \mathrm{P}(t)\mathrm{y} + \mathrm{g}(t) \tag{2}$$

where

$$\mathrm{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \dots \\ y_n(t) \end{pmatrix}, \ \mathrm{P} = (p_{ij}(t)), \ \mathrm{g} = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \dots \\ g_n(t) \end{pmatrix}$$

Continued

- Likewise, a homogeneous linear system can be written as
$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y} \qquad (3)$$
- There will be several solutions of (2) or of (3).
Each solution is a vector functions. We denote them by
$$\mathbf{y}^{(1)}(t) = \begin{pmatrix} y_{11}(t) \\ y_{21}(t) \\ \cdots \\ y_{n1}(t) \end{pmatrix}, \cdots, \mathbf{y}^{(k)}(t) = \begin{pmatrix} y_{1k}(t) \\ y_{2k}(t) \\ \cdots \\ y_{nk}(t) \end{pmatrix}.$$

Principle of superposition

- Lemma 5.4.1: Suppose $\mathrm{y}^{(1)}, \ldots, \mathrm{y}^{(k)}$ are solutions of a homogeneous linear system (3).
  Then any constant linear combination
  $$\mathrm{y} = c_1 \mathrm{y}^{(1)} + \cdots + c_k \mathrm{y}^{(k)} \tag{4}$$
  is also a solution of the same system (3).

The converse of the Principle of superposition is also true, in the following sense.

Converse of Principle of superposition

Theorem 5.4.2: Suppose $\mathrm{y}^{(1)}, \ldots, \mathrm{y}^{(n)}$ are solution of a homogeneous linear system (3). Let $\mathrm{Y}(t) = \begin{pmatrix} \mathrm{y}^{(1)} & \ldots & \mathrm{y}^{(n)} \end{pmatrix}$. Define the Wronskian

$$W(t) := W(\mathrm{y}^{(1)}, \ldots, \mathrm{y}^{(n)})(t) = |\mathrm{Y}(t)|$$

$$\text{Assume,} \qquad W(t) \neq 0 \qquad \forall\ t \in (\alpha, \beta)$$

(*equivalently,* $\mathrm{y}^{(1)}(t), \ldots, \mathrm{y}^{(n)}(t)$ *are linearly independent*)

Let $\mathrm{y} = \varphi(\mathrm{t})$ be any solution of (3). Then

$$\mathrm{y} = \varphi(\mathrm{t}) = c_1 \mathrm{y}^{(1)} + \cdots + c_n \mathrm{y}^{(n)} \quad \text{for some } c_1, \cdots, c_n \in \mathbb{R}.$$

Continued

- **Definition.** In the above case, we say that

$$\mathrm{y}^{(1)}(t), \ldots, \mathrm{y}^{(n)}(t)$$

form a Fundamental Set of Solutions of (3).