Chapter 4: Higher Order ODE
§4.2 Linear Homogeneous ODE with constant coefficients

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Goals

In this section we give an overview of Linear Homogeneous ODE, with constant coeffients. Again, the main point of this section is that the methods of solving such ODE is strikingly similar to that of 2^{nd}-order Homogeneous Linear ODE.

Definition

Recall the following definition.
Definition A Homogeneous Linear ODE is said to have constant coefficient looks like

$$\mathcal{L}(y) = a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0 \qquad (1)$$

with $a_0, a_1, \cdots, a_n \in \mathbb{R}$ and $a_n \neq 0$.

The Characteristic equation

As in the case of 2^{nd}-order,
an exponential functions $y = e^{rt}$ will be a solutions of (1),
for some real or complex number r; checked as follows.

- Substituting $y = e^{rt}$ in (1) we get

$$\mathcal{L}(e^{rt}) = e^{rt}\left(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0\right) = 0$$

- It follows, $y = e^{rt}$ is a solution of (1) if and only if

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0 \qquad (2)$$

Continued

- The Equation (2) is called the characteristic equation (CE) of (1). So, solving the ODE (1) reduces to solving the CE.
- The polynomial

$$\rho(r) := a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 \qquad (3)$$

is the characteristic polynomial of (1). So, the characteristic equation can be written as

$$\rho(r) = 0$$

The Roots of the characteristic polynomial

We can write

$$\rho(r) = \begin{cases} = (r-\lambda_1)(r-\lambda_2)\cdots(r-\lambda_n) & \lambda_i \in \mathbb{C} \\ = (r-r_1)^{k_1}(r-r_2)^{k_2}\cdots(r-r_m)^{k_m} & k_i \geq 1, r_i \in \mathbb{C} \end{cases}$$

$k_1 + \cdots + k_m = n$, where $r_1, \ldots, r_m \in \mathbb{C}$ are distinct

Continued: The Fundamental Theorem

- *For this frame, assume ρ has real or complex coefficients.*
- The above is know as *Fundamental Theorem of Algebra*. This is a very deep and important theorem.
- We can read: any polynomial $\rho(r)$ with $\deg(\rho) = n$, **has n roots** (complex), counting multiplicity. (k_i (in the last frame) is called the multiplicity of r_i.)
- **Contrast:** We cannot say the same about real roots.

Solutions of (1): Real Root

If r_1 is real, then r_1 spits out the following k_1 solutions of (1):

$$\begin{cases} y = e^{r_1 t} \\ y = te^{r_1 t} \\ y = t^2 e^{r_1 t} \\ \cdots \\ y = t^{k_1 - 1} e^{r_1 t} \end{cases}$$

Likewise, for any real root r_i.

Continued

If r_1 is complex (i.e. $r_1 \notin \mathbb{R}$), then its conjugate $\overline{r_1}$ is also a root of $\rho(r)$. Without loss of generality $r_2 = \overline{r_1}$. The pair $\begin{cases} r_1 = \lambda_1 + \mu_1 i \\ \overline{r_1} = r_2 = \lambda_1 - \mu_2 i \end{cases}$ spits out $2k_1$ solutions of (1):

$$\begin{cases} y = e^{\lambda_1 t} \cos \mu_1 t & y = e^{\lambda_1 t} \sin \mu_1 t \\ y = te^{\lambda_1 t} \cos \mu_1 t & y = te^{\lambda_1 t} \sin \mu_1 t \\ y = t^2 e^{\lambda_1 t} \cos \mu_1 t & y = t^2 e^{\lambda_1 t} \sin \mu_1 t \\ \cdots & \cdots \\ y = t^{k_1 - 1} e^{\lambda_1 t} \cos \mu_1 t & y = t^{k_1 - 1} e^{\lambda_1 t} \sin \mu_1 t \end{cases}$$

Likewise, for each pair of complex roots $r_i, \overline{r}_i$ of $\rho(r)$.

A Fundamental Set and General Solutions

The process explained above, give total of n real solutions (1):

$$\begin{cases} y = y_1(t), \\ y_2 = y_2(t), \\ \cdots, \\ y = y_n(t) \end{cases}$$

Theorem 4.2.1 The list of n solutions above form a Fundamental Set of Solutions of (1).
So, a general solution of (1) is:

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \qquad \text{where} \quad c_i \in \mathbb{R} \qquad (4)$$

Solving Some Examples

Unlike quadratic formula,
for polynomials $\rho(r)$, with $\deg(\rho(r)) \geq 3$
there no straight forward formula to find the roots.
We consider a few simple problems.

Example 1

Find the general solution of homogeneous ODE
$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$

- The CE: $r^3 - 2r^2 - r + 2 = 0.$
  $(r+1)(r-1)(r-2) = 0.$ So, $r_1 = -1,\ r_2 = 1,\ r_3 = 2.$
- By (4) the general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + c_3 e^{r_3 t} = c_1 e^{-t} + c_2 e^{t} + c_3 e^{2t}$$

Example 2

Find the general solution of homogeneous ODE
$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

- The CE: $r^3 - 2r^2 + r - 2 = 0$.
  $(r^2+1)(r-2) = 0$. So, $r_1 = 2$, $r_2 = i$, $r_3 = -i$.
- $r_1 = 2$ contributes a solution $y_1 = e^{r_1 t} = e^{2t}$.
  The pair of complex root $r_2 = i$, $r_2 = -i$ contributes two solution $\begin{cases} y_2 = \cos t \\ y_3 = \sin t \end{cases}$
- By (4) the general solution is

$$y = c_1y_1 + c_2y_2 + c_3y_3 = c_1e^{2t} + c_2\cos t + c_3\sin t$$

Example 3

Find the general solution of homogeneous ODE
$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5y = 0$

- The CE: $r^3 - r^2 + 3r + 5 = 0$. We see $r = -1$ is a root $(r+1)(r^2 - 2r + 5) = 0$. So, $r_1 = -1, r_2, r_3 = 1 \pm 2i$.
- $r_1 = -1$ contributes a solution $y_1 = e^{r_1 t} = e^{-t}$.
  The pair $r_2, r_3 = 1 \pm 2i$ contributes two solution
  $\begin{cases} y_2 = e^t \cos 2t \\ y_3 = e^t \sin 2t \end{cases}$
- By (4) the general solution is

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 e^{-t} + c_2 e^t \cos 2t + c_3 e^t \sin 2t$$

Example 4

Find the general solution of homogeneous ODE
$\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3} + 9\frac{d^2y}{dx^2} + 16\frac{dy}{dx} + 20y = 0$

- The CE: $r^4 + 4r^3 + 9r^2 + 16r + 20 = 0.$
  $(r^2 + 4)(r^2 + 4r + 5) = 0$. So,
  $$r_1, r_2 = \pm 2i, r_3, r_4 = -2 \pm i$$
- $r_1, r_2 = \pm 2i$ contributes a two solution $\begin{cases} y_1 = \cos 2t \\ y_2 = \sin 2t \end{cases}$
  The pair $r_3, r_4 = -2 \pm i$ contributes two solution
  $\begin{cases} y_3 = e^{-2t}\cos t \\ y_4 = e^{-2t}\sin t \end{cases}$

Continued

- By (4) the general solution is

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$$

$$= c_1 \cos 2t + c_2 \sin 2t + c_3 e^{-2t} \cos t + + c_4 e^{-2t} \sin t$$