

# Chapter 6: The Laplace Transform

## §6.4 Systems with Discontinuous Functions

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# Discontinuous Force Functions

- ▶ As in §6.2, in this section we solve systems of nonhomogeneous linear second order with constant coefficients, using Laplace Transform and Inverse Laplace Transforms.
- ▶ However, the nonhomogeneous part  $g(t)$  would be **discontinuous**, with an eye for the Dirac  $\delta$  function and  $d_\tau(t - c)$ .
- ▶ As in §6.2, usually we would use the charts of Laplace Transform and Inverse Laplace Transforms.

# Example 1

Solve the IVP

$$y'' + 9y = f(t) = \begin{cases} 1 & \pi \leq t < 3\pi \\ 0 & 3\pi \leq t < \infty \end{cases} \quad y(0) = 0, y'(0) = 1.$$

- ▶ We write  $f(t) = u_{\pi}(t) - u_{3\pi}(t)$ .
- ▶ So, we have  $y'' + 9y = u_{\pi}(t) - u_{3\pi}(t)$
- ▶ We write  $Y(s) = \mathcal{L}\{y\}$ . So,

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{u_{\pi}\} - \mathcal{L}\{u_{3\pi}\} \implies$$

$$(s^2 Y - sy(0) - y'(0)) + 9Y = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s} \Rightarrow$$

$$(s^2 + 9)Y = 1 + \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s} \Rightarrow$$

$$Y = \frac{1}{s^2 + 1} + \frac{e^{-\pi s}}{s(s^2 + 1)} - \frac{e^{-3\pi s}}{s(s^2 + 1)}$$

$$= \mathcal{L}\{\sin t\} + (e^{-\pi s} - e^{-3\pi s}) \left( \frac{a}{s} + \frac{bs + c}{s^2 + 1} \right)$$

$$= \mathcal{L}\{\sin t\} + (e^{-\pi s} - e^{-3\pi s}) \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right)$$

## Continued

$$Y = \mathcal{L}\{\sin t\} + \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s} + \frac{se^{-\pi s}}{s^2 + 1} - \frac{se^{-3\pi s}}{s^2 + 1}$$

$$= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_\pi\} - \mathcal{L}\{u_{3\pi}\} + e^{-\pi s}\mathcal{L}\{\cos t\} - e^{-3\pi s}\mathcal{L}\{\cos t\}$$

$$= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_\pi\} - \mathcal{L}\{u_{3\pi}\} + \mathcal{L}\{u_\pi \cos(t-\pi)\} - \mathcal{L}\{u_{3\pi} \cos(t-3\pi)\}$$

$$= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_\pi\} - \mathcal{L}\{u_{3\pi}\} - \mathcal{L}\{u_\pi \cos(t)\} + \mathcal{L}\{u_{3\pi} \cos(t)\}$$

$$\text{So } Y = \mathcal{L}\{y\} = \mathcal{L}\{\sin t + u_\pi - u_{3\pi} - u_\pi \cos(t) + u_{3\pi} \cos(t)\}$$

$$\text{So } y = \sin t + u_\pi - u_{3\pi} - u_\pi \cos(t) + u_{3\pi} \cos(t)$$

# Example 2

Solve the IVP

$$y'' + y' - 6y = u_2(t) \quad y(0) = 0, y'(0) = 1.$$

- ▶ We write  $Y(s) = \mathcal{L}\{y\}$ . There are two steps:
  - ▶ Compute  $Y(s)$ , by application of Laplace transform  $\mathcal{L}$ .
  - ▶ Compute  $y = \mathcal{L}^{-1}\{Y(s)\}$  by application of Inverse Laplace transform  $\mathcal{L}$ .

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = \mathcal{L}\{u_2(t)\}$$

$$(s^2 Y(s) - sy(0) - y'(0)) + (sY(s) - y(0)) - 6Y(s) = \frac{1}{s}e^{-2s}$$

$$(s^2 Y(s) - 1) + sY(s) - 6Y(s) = \frac{1}{s}e^{-2s}$$

$$Y(s) = \frac{1}{s(s^2 + s - 6)}e^{-2s} + \frac{1}{s^2 + s - 6}$$

$$= \frac{1}{s(s^2 + 3s + 2)}e^{-2s} + \frac{1}{s-2} - \frac{1}{s+3}$$

$$= \frac{1}{s(s^2 + 3s + 2)}e^{-2s} + \mathcal{L}\{e^{2t}\} - \mathcal{L}\{e^{-3t}\}$$

## Continued

$$\begin{aligned}\text{So, } Y &= \frac{1}{s(s+3)(s-2)} e^{-2s} + \mathcal{L}\{e^{2t} - e^{-3t}\} \\ &= e^{-2s} \left( \frac{a}{s} + \frac{b}{s-2} + \frac{c}{s+3} \right) + \mathcal{L}\{e^{2t} - e^{-3t}\} \\ &= e^{-2s} \left( -\frac{1}{6} \frac{1}{s} + \frac{1}{10} \frac{1}{s-2} + \frac{1}{15} \frac{1}{s+3} \right) + \mathcal{L}\{e^{2t} - e^{-3t}\} \\ &= e^{-2s} \left( -\frac{1}{6} \mathcal{L}\{1\} + \frac{1}{10} \mathcal{L}\{e^{2t}\} + \frac{1}{15} \mathcal{L}\{e^{-3t}\} \right) + \mathcal{L}\{e^{2t} - e^{-3t}\} \\ &= e^{-2s} \left( \mathcal{L} \left\{ -\frac{1}{6} + \frac{1}{10} e^{2t} + \frac{1}{15} e^{-3t} \right\} \right) + \mathcal{L}\{e^{2t} - e^{-3t}\}\end{aligned}$$

## Continued

$$\begin{aligned}\text{So, } Y &= e^{-2s} \left( \mathcal{L} \left\{ \frac{-5 + 3e^{2t} + 2e^{-3t}}{30} \right\} \right) + \mathcal{L}\{e^{2t} - e^{-3t}\} \\&= \left( \mathcal{L} \left\{ u_2(t) \frac{-5 + 3e^{2(t-2)} + 2e^{-3(t-2)}}{30} \right\} \right) + \mathcal{L}\{e^{2t} - e^{-3t}\} \Rightarrow \\ \mathcal{L}\{y\} = Y &= \mathcal{L} \left\{ u_2(t) \frac{-5 + 3e^{2(t-2)} + 2e^{-3(t-2)}}{30} + e^{2t} - e^{-3t} \right\} \\ \text{So, } y &= u_2(t) \frac{-5 + 3e^{2(t-2)} + 2e^{-3(t-2)}}{30} + e^{2t} - e^{-3t}\end{aligned}$$

# Example 3

Solve the IVP

$$y'' + 9y = g(t) = \begin{cases} t & 0 \leq t < 3 \\ 3 & 3 \leq t \end{cases} \quad y(0) = 0, y'(0) = 1.$$

- ▶ We write  $Y(s) = \mathcal{L}\{y\}$ . There are two steps:
  - ▶ Compute  $Y(s)$ , by application of Laplace transform  $\mathcal{L}$ .
  - ▶ Compute  $y = \mathcal{L}^{-1}\{Y(s)\}$  by application of Inverse Laplace transform  $\mathcal{L}$ .

- ▶ Write  $g(t) = (1 - u_3(t))t + 3u_3(t) = t - u_3(t)(t - 3)$
- ▶ From the Charts So,

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - e^{-3s} \frac{1}{s^2}$$

- ▶ Apply  $\mathcal{L}$  the System:

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{g(t)\} = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}$$

$$s^2 Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{1}{s^2} - e^{-3s} \frac{1}{s^2}$$


$$s^2 Y(s) - 1 + 9Y(s) = \frac{1}{s^2} - e^{-3s} \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2 + 9} + \frac{1}{s^2(s^2 + 9)} - e^{-3s} \frac{1}{s^2(s^2 + 9)}$$

$$= \frac{1}{s^2 + 9} + \frac{1}{9} \frac{1}{s^2} - \frac{1}{9} \frac{1}{s^2 + 9} - e^{-3s} \frac{1}{s^2(s^2 + 9)}$$

$$= \frac{1}{9} \frac{1}{s^2} + \frac{8}{9} \frac{1}{s^2 + 9} - e^{-3s} \frac{1}{s^2(s^2 + 9)}$$

$$= \frac{1}{9} \mathcal{L}\{t\} + \frac{8}{27} \mathcal{L}\{\sin 3t\} - e^{-3s} \frac{1}{s^2(s^2 + 9)}$$

# Continued

So,

$$\begin{aligned} Y &= \mathcal{L} \left\{ \frac{3t + 8 \sin 3t}{27} \right\} - e^{-3s} \left( \frac{1}{9} \frac{1}{s^2} - \frac{1}{9} \frac{1}{s^2 + 9} \right) \\ &= \mathcal{L} \left\{ \frac{3t + 8 \sin 3t}{27} \right\} - e^{-3s} \left( \frac{1}{9} \mathcal{L}\{t\} - \frac{1}{27} \mathcal{L}\{\sin 3t\} \right) \\ &= \mathcal{L} \left\{ \frac{3t + 8 \sin 3t}{27} \right\} - e^{-3s} \mathcal{L} \left\{ \frac{3t - \sin 3t}{27} \right\} \\ &= \mathcal{L} \left\{ \frac{3t + 8 \sin 3t}{27} \right\} - \mathcal{L} \left\{ u_3(t) \frac{3(t-3) - \sin 3(t-3)}{27} \right\} \end{aligned}$$

# Continued

So,  $\mathcal{L}\{y\} = Y$

$$= \mathcal{L} \left\{ \frac{3t + 8 \sin 3t}{27} - u_3(t) \frac{3(t-3) - \sin 3(t-3)}{27} \right\}$$

So,

$$y = \frac{3t + 8 \sin 3t}{27} - u_3(t) \frac{3(t-3) - \sin 3(t-3)}{27}$$