Chapter 5: System of 1^{st}-Order Linear ODE §5.8 Nonhomogeneous Linear Systems

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Nonhomogeneous Linear Systems

Finally, we consider Nonhomogeneous Linear Systems.

- A nonhomogeneous linear system can be written as:

$$\mathrm{y}' = \mathrm{P}(t)\mathrm{y} + \mathrm{g}(t) \tag{1}$$

where $\mathrm{P}(t) = (p_{ij}(t))$ is an $n \times n$-matrix,

$$\mathrm{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \cdots \\ g_n(t) \end{pmatrix}$$ is a column matrix.

- We assume that $p_{ij}(t), g_j(t)$ are all continuous functions on an open interval $I : \alpha < t < \beta$.

General Solutions

- Corresponding to (1), we have the homogeneous system:

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y} \tag{2}$$

- As in Chapter 3, 4, and in Linear Algebra, a general solution of a non homogeneous system (1) has the form:

$$\mathbf{y} = \mathbf{Y} + \mathbf{y}_c \qquad \text{where} \tag{3}$$

  - $\mathbf{Y} = \mathbf{Y}(t)$ is a particular solution of the system (1),
  - $\mathbf{y}_c$ is the general solution of the homogeneous system (2), which can be computed using methods in §5.5, 5.6, 5.7.

Methods to Find a particular solution Y

The following are some of the possible methods to compute a particular solution Y:

- Diagonalization.
- Method of Undetermined coefficients.
- Variation of parameters.
- Laplace transforms (extension of chapter 6)

We would only discuss the first one. Further, we would only consider the case, when $P(t) = A$ is a constant matrix.

Diagonalizable system

Consider nonhomogeneous systems:

$$\text{with constant coefficients} \quad y' = Ay + g(t) \tag{4}$$

where A is an $n \times n$-matrix with constant entries, $g(t)$ is as in (1). The corresponding homogeneous system:

$$y' = Ay \tag{5}$$

- Sometimes, the matrix A would be diagonalizable. This means, there is an invertible matrix T such that

$$T^{-1}AT = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & r_n \end{pmatrix} =: D \qquad (6)$$

  is a diagonal matrix. This would be the case when A has a set of n linearly independent eigen VECTORS.
- It also follows $AT = TD$
- In this case, $r_1, r_2, \ldots, r_n$ would be the eigenvalues of A and i^{th}-column of T would be the eigenvector for r_i.

- Most importantly, we change variables:
$$\mathbf{z} = \mathbf{T}^{-1}\mathbf{y} \Longrightarrow \mathbf{z}' = \mathbf{T}^{-1}\mathbf{y}' = \mathbf{T}^{-1}(\mathbf{A}\mathbf{y} + \mathbf{g}(t)) \Longrightarrow$$
$$\mathbf{z}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \mathbf{D}\mathbf{T}^{-1}\mathbf{y} + \mathbf{h}(t) = \mathbf{D}\mathbf{z} + \mathbf{h}(t)$$
where $\mathbf{h}(t) := \mathbf{T}^{-1}\mathbf{g}(t)$ is a column vector of functions.
- It follows $\mathbf{z}' = \mathbf{D}\mathbf{z} + \mathbf{h}(t) \Longrightarrow$
$$\begin{pmatrix} z_1' \\ z_2' \\ \cdots \\ z_n' \end{pmatrix} = \begin{pmatrix} r_1 z_1 + h_2(t) \\ r_2 z_2 + h_2(t) \\ \cdots \\ r_n z_n + h_n(t) \end{pmatrix} \qquad (7)$$

- For future reference:

$$\begin{pmatrix} z_1' - r_1 z_1 \\ z_2' - r_2 z_2 \\ \cdots \\ z_n' - r_n z_n \end{pmatrix} = \mathrm{T}^{-1}\mathrm{g}(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \\ \cdots \\ h_n(t) \end{pmatrix} \qquad (8)$$

- So, for $i = 1, 2, \ldots, n$, $x_i' = r_i z_i + h_i(t)$ are 1^{st}-order Linear ODE in one variable (see §2.1 or the Appendix below).

- By see (14) below (from §2.1) the general solution for z_i:

$$z_i = e^{r_i t}\left[\int e^{-r_i t} h_i(t) + c_i\right] = e^{r_i t}\left[\int_{t_0}^{t} e^{-r_i s} h_i(s) + c_i\right]$$

- We need only a solution of (4). So, take $c_i = 0$:

$$z_i = e^{r_i t}\left[\int e^{-r_i t} h_i(t)\right] \tag{9}$$

- Clarification: The constant c_i, will get absorbed in y_c term of the general solution $\mathrm{y} = \mathrm{Y} + \mathrm{y}_c$ of (4).
- Now, to solve (4), compute $\mathrm{y} = \mathrm{Tz}$.

Compute T

- Compute the eigenvalues $r_1, r_2, \ldots, r_n$ by solving $|A - rI| = 0$. Some of these r_i may repeat. (Write them in increasing order.)
- Assume A has a set of n linearly independent eigenvectors $\xi_1, \xi_2, \cdots, \xi_n$.
- Let, $T = \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \end{pmatrix}$.
- Compute T^{-1} (Use TI-84, unless it gives clumsy output).
- We would be considering problems, with $n = 2, 3$. We would also avoid complex eigenvalues.

Example 1

Find the general solution of

$$y' = \begin{pmatrix} 2 & 2 \\ -2 & -3 \end{pmatrix} y + \begin{pmatrix} -e^t \\ -e^{-t} \end{pmatrix} \tag{10}$$

- The corresponding homogeneous equation

$$y' = \begin{pmatrix} 2 & 2 \\ -2 & -3 \end{pmatrix} y \tag{11}$$

- We can use the above method, only if there are 2 linearly independent eigenvectors. In particular, if all the eigenvalues are distinct.
- Eigenvalues of A, are given by

$$\begin{vmatrix} 2-r & 2 \\ -2 & -3-r \end{vmatrix} = 0 \Longrightarrow \begin{cases} (2-r)(-3-r)+4=0 \\ \text{So}, \ r=-2,1 \end{cases}$$

- Since two eigenvalues are distinct, there will be two linearly independent eigenvectors.

Eigenvectors

- Eigenvectors for $r = -2$ is given by $(A - rI)\xi = 0$:

$$\begin{pmatrix} 2+2 & 2 \\ -2 & -3+2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{cases} 0 = 0 \\ 2\xi_1 + \xi_2 = 0 \end{cases} \quad \text{With } \xi_1 = 1, \quad \xi^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

is an eigenvector for $r = -2$.

- Corresponding solution for the homogeneous ODE (11):

$$y^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$$

Eigenvectors

- Eigenvectors for $r = 1$ is given by $(A - rI)\xi = 0$:

$$\begin{pmatrix} 2-1 & 2 \\ -2 & -3-1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{cases} 0 = 0 \\ \xi_1 + 2\xi_2 = 0 \end{cases} \quad \text{With } \xi_2 = 1, \quad \xi^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

is an eigenvector for $r = 1$.

- Corresponding solution for the homogeneous ODE (11):

$$\mathrm{y}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{t}$$

The Matrix T

- The matrix T is

$$T = \begin{pmatrix} \xi^{(1)} & \xi^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$

- Also

$$T^{-1} = \frac{1}{|T|}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \frac{1}{-3}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Change variable $z = T^{-1}y$

- We change variables $z = T^{-1}y$. By (8)

$$\begin{pmatrix} z_1' - r_1 z_1 \\ z_2' - r_2 z_1 \end{pmatrix} = T^{-1}g(t) = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{pmatrix} g(t)$$

$$\begin{pmatrix} z_1' + 2z_1 \\ z_2' - z_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} -e^t \\ -e^{-t} \end{pmatrix}$$

$$\begin{pmatrix} z_1' + 2z_1 \\ z_2' - z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}e^t + \frac{2}{3}e^{-t} \\ \frac{2}{3}e^t + \frac{1}{3}e^{-t} \end{pmatrix}$$

Solve for z_1

- We have $z_1' + 2z_1 = \frac{1}{3}e^t + \frac{2}{3}e^{-t}$
- The IF $\mu(t) = \exp(\int 2dt) = e^{2t}$
- By (14) a solution for y_1:

$$z_1 = \frac{1}{\mu(t)}\left[\int \mu(t)h_1(t)dt\right]$$

$$= e^{-2t}\left[\int e^{2t}\left(\frac{1}{3}e^t + \frac{2}{3}e^{-t}\right)dt\right]$$

$$= e^{-2t}\left[\left(\frac{1}{9}e^{3t} + \frac{2}{3}e^t\right)\right] = \frac{1}{9}e^t + \frac{2}{3}e^{-t}$$

Solve for z_2

- We have $z_2' - z_2 = \frac{2}{3}e^t + \frac{1}{3}e^{-t}$
- The IF $\mu(t) = \exp(\int -dt) = e^{-t}$
- By (14), a solution z_2:

$$z_2 = \frac{1}{\mu(t)}\left[\int \mu(t)h_2(t)dt\right]$$

$$= e^t\left[\int e^{-t}\left(\frac{2}{3}e^t + \frac{1}{3}e^{-t}\right)dt\right]$$

$$= e^t\left[\left(\frac{2}{3}t - \frac{1}{6}e^{-2t}\right)\right] = \frac{2}{3}te^t - \frac{1}{6}e^{-t}$$

A solution Y of (10)

- So,

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{9}e^t + \frac{2}{3}e^{-t} \\ \frac{2}{3}te^t - \frac{1}{6}e^{-t} \end{pmatrix}$$

- Finally, a particular solution of (10):

$$Y = Tz = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{9}e^t + \frac{2}{3}e^{-t} \\ \frac{2}{3}te^t - \frac{1}{6}e^{-t} \end{pmatrix}$$

(*I would leave it in this matrix form.*)

The general solution of (10):

- The general solution (10):

$$\mathrm{y} = \mathrm{Y} + \mathrm{y}_c = \mathrm{Y} + c_1 \mathrm{y}^{(1)} + c_2 \mathrm{y}^{(2)}$$

(*Again, I would leave it in this matrix form,* where Y, $\mathrm{y}^{(1)}$, $\mathrm{y}^{(2)}$ are given above.)

Example 2

Find the general solution of

$$\mathrm{y}' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathrm{y} + \begin{pmatrix} -e^t \\ 2e^t \end{pmatrix} \qquad (12)$$

- The corresponding homogeneous equation

$$\mathrm{y}' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathrm{y} \qquad (13)$$

- Eigenvalues of A, are given by

$$\begin{vmatrix} 1-r & 4 \\ 1 & 1-r \end{vmatrix} = 0 \Longrightarrow r^2 - 2r - 3 = 0 \Longrightarrow \ r = -1, 3$$

- Since the eigenvalues are distinct, the corresponding eigen vectors would be linearly independent. So, we can use the method above.

Eigenvectors

- Eigenvectors for $r = -1$ is given by $(A - rI)\xi = 0$:

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{cases} 0 = 0 \\ \xi_1 + 2\xi_2 = 0 \end{cases} \quad \text{With } \xi_2 = 1, \quad \xi^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

is an eigenvector for $r = -1$.

- Corresponding solution for the homogeneous ODE (13):

$$\xi^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t}$$

Eigenvectors

- Eigenvectors for $r = 3$ is given by $(A - rI)\xi = 0$:

$$\begin{pmatrix} 1-3 & 4 \\ 1 & 1-3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The second row is a multiple of the first row. It follows:

$$\begin{cases} 0 = 0 \\ \xi_1 - 2\xi_2 = 0 \end{cases} \quad \text{With } \xi_2 = 1, \quad \xi^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is an eigenvector for $r = 3$.

- Corresponding solution for the homogeneous ODE (13):

$$\mathbf{x}^{(2)} = \xi^{(2)} e^{rt} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t}$$

The Matrix T

- The matrix T is

$$T = \begin{pmatrix} \xi^{(1)} & \xi^{(2)} \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$$

- Also

$$T^{-1} = \frac{1}{|T|} \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 1 & -2 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

(*I am OK with terminating decimal numbers, because there is no rounding involved.*)

Change variable $z = T^{-1}y$

- We change variables $z = T^{-1}y$. By (8)

$$\begin{pmatrix} z_1' - r_1 z_1 \\ z_2' - r_2 z_1 \end{pmatrix} = T^{-1}g(t) = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} g(t)$$

$$\begin{pmatrix} z_1' + z_1 \\ z_2' - 3z_1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -e^t \\ 2e^t \end{pmatrix} = \begin{pmatrix} \frac{5}{4}e^t \\ \frac{3}{4}e^t \end{pmatrix}$$

Solve for z_1

- We have $z_1' + z_1 = \frac{5}{4}e^t$
- The IF $\mu(t) = \exp(\int dt) = e^t$
- By (14) a solution for z_1:

$$z_1 = \frac{1}{\mu(t)}\left[\int \mu(t)h_1(t)dt\right] = e^{-t}\left[\int e^t\left(\frac{5}{4}e^t\right)dt\right]$$

$$= e^{-t}\left[\frac{5}{4}\int e^{2t}dt\right] = e^{-t}\left[\frac{5}{4}\frac{e^{2t}}{2}\right] = \frac{5}{8}e^t$$

Solve for z_2

- We have $z_2 - 3z_2 = \frac{3}{4}e^t$
- The IF $\mu(t) = \exp(\int -3dt) = e^{-3t}$
- By (14), a solution z_2:

$$z_2 = \frac{1}{\mu(t)}\left[\int \mu(t)h_2(t)dt\right] = e^{3t}\left[\int e^{-3t}\left(\frac{3}{4}e^t\right)dt\right]$$

$$= e^{3t}\left[\frac{3}{4}\int e^{-2t}dt\right] = \frac{3}{4}e^{3t}\frac{e^{-2t}}{-2} = -\frac{3}{8}e^t$$

A Particualr solution Y of (12)

- So,

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{5}{8}e^t \\ -\frac{3}{8}e^t \end{pmatrix}$$

- Finally, a particualr solution of (12):

$$Y = Tz = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{8}e^t \\ -\frac{3}{8}e^t \end{pmatrix} = \begin{pmatrix} -2e^t \\ \frac{1}{4}e^t \end{pmatrix}$$

The general solution of (12):

- The general solution (12):

$$\mathrm{y} = \mathrm{Y} + \mathrm{y}_c = \mathrm{Y} + c_1\mathrm{y}^{(1)} + c_2\mathrm{y}^{(2)}$$

where Y, $\mathrm{y}^{(1)}$, $\mathrm{y}^{(2)}$ are given above.

The Solution of FOLE

- Recall a 1^{st}-order Linear ODE had the form $y' + p(t)y = g(t)$.
- The integrating factor: $\mu(t) = \exp\left(\int p(t)dt\right)$
- The general solution:

$$y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + c\right].$$

- Witn $c = 0$, a solution for y is:

$$y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt\right] \tag{14}$$