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The Homotopy obstructions in complete intersections

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Abstract.

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1. Introduction

For (smooth) affine schemes $X = \mathrm{Spec}\,(A)$ over algebraically closed fields k and for projective A-modules P, with $\dim X = d = rank(P)$, N. Mohan Kumar and M. P. Murthy considered top Chern classes $C^d(P) \in CH^d(X)$, in the Chow Group of zero cycles, as obstruction for P to split off a free direct summand ([MoM,Mk1,Mk2,Mu,MMu]). In deed, the results in [Mu,MMu], were fairly finalistic. For an ideal I, an obstruction class $\zeta(P, I) \in CH^d(X)$ was written down [MMu]. It was established that there is a surjective map $P \twoheadrightarrow I$ if and only if $\zeta(P, I) = 0$ and there is a surjective map $P \twoheadrightarrow \frac{I}{I^2}$.

Subsequent to that, based on some Homotopy Relations (see Lemma 2.3), Madhav V. Nori (around 1990) laid out a set of ideas to deal with the questions of such obstructions in broader contexts, like when X is a regular or a noetherian affine scheme. These were communicated verbally to some in a very informal and open ended manner. Because of the nature of these communications, not everyone heard the same thing and these ideas took the form of some folklores. As a result, versions of this set of ideas available (or not) in the literature (e.g. [M3,MV,MS,BS1,BS2,BS3,BK]) have been up to the interpretations and adaptations by the recipients of these communications, much to their credit, and the stated hypotheses may differ. Because of the openendedness and broadness of these ideas, they appeared to be more like a research Program (*the Homotopy Program*) to some, which is how we would refer to the same in this article. Sometimes it may even be difficult to say whether certain part of the program was actually explicitly articulated by Nori or were part of the adaptations by others. There is no systematic exposition of this program available in the literature and certain aspects failed to receive deserved traction. Nori never classified these as conjectures or otherwise. However, some results followed too quickly [M3], to treat them as anything less than conjectures, subject to further fine tuning.

Analogy to the Obstruction Theory for vector bundles ([St]) was the main backdrop behind this program and central to this Program was the Homotopy conjecture of Nori. The following is the statement of the Homotopy Conjecture from [M3], which would most likely be an adaptation by the respective author.

Conjecture 1.1 (Homotopy Conjecture). *Suppose* $X = \mathrm{Spec}\,(A)$ *is a smooth affine variety, with* $\dim X = d$. *Let* P *be a projective* A*-module of rank* r *and* $f_0 : P \twoheadrightarrow I$ *be a surjective homomorphism, onto an ideal* I *of* A.

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Assume $Y = V(I)$ is smooth with $\dim Y = d - r$. Also suppose $Z = V(J) \subseteq \mathrm{Spec}\,(A[T]) = X \times \mathbb{A}^1$ is a smooth subscheme, such that Z intersects $X \times 0$ transversally in $Y \times 0$. Now, suppose that $\varphi : P[T] \twoheadrightarrow \frac{J}{J^2}$ is a surjective map such that $\varphi_{|T=0} = f_0 \otimes \frac{A}{I}$. Then, there is a surjective map $F : P[T] \twoheadrightarrow J$ such that (i) $F_{|T=0} = f_0$ and (ii) $F_{|Z} = \varphi$.

The best result, up to date, on this Conjecture 1.1 is due to Bhatwadekar and Keshari [BK]. While the Conjecture 1.1 would fail without the regularity hypothesis [BS1, Example 6.4], existing results (see [M3,BS1,BK]) indicate that with suitable hypotheses the regularity and/or transversality hypotheses may be spared. However, the Conjecture 1.1, as stated, would even fail in some cases when A is regular (see [BS1, Example 3.15]), which are, conjecturally, the exceptions. In analogy to the obstructions $\zeta(P, I) \in CH^d(X)$ mentioned above ([MMu]), the objective of the Homotopy Conjecture 1.1 was to detect, for an ideal I, when a surjective map $f : P \twoheadrightarrow \frac{I}{I^2}$ would lift to a surjective map $F : P \twoheadrightarrow I$, using the homotopy relations given by surjective maps $\varphi : P[T] \twoheadrightarrow \frac{J}{J^2}$, where J is an ideal in $A[T]$ (see Lemma 2.3).

In this article, we will discuss the Homotopy Program, only in the complete intersections case, that is when $P = A^n$ is free, with $n \geq 2$. To clarify the Homotopy Obstruction set of Nori, in this complete intersection case, let $\mathcal{L}O(A, n)$ denote the set of all pairs (I, ω), where I is an ideal of A and $\omega : A^n \twoheadrightarrow \frac{I}{I^2}$ is a surjective homomorphism. By substituting $T = 0, 1$, we obtain two maps $\mathcal{L}O(A, n) \xleftarrow{T=0} \mathcal{L}O(A[T], n) \xrightarrow{T=1} \mathcal{L}O(A, n)$. This leads to an equivalence relation (using chains) on $\mathcal{L}O(A, n)$ and a set of all equivalence classes $\pi_0(\mathcal{L}O(A, n))$. In the recent past, a similar Homotopy Obstruction set (pre-sheaf), $\pi_0(Q_{2n})(A)$ was considered in [F1], to serve the same purpose. We clarify (see Lemma 2.3) that $\pi_0(Q_{2n})(A)$ coincides with $\pi_0(\mathcal{L}O(A, n))$. This puts some of the developments [F1,M2] (*now retracted*) in the recent past, in the framework of the Homotopy Program of Nori. In deed, in this article, we mostly investigate the structure of this obstruction set $\pi_0(\mathcal{L}O(A, n)) = \pi_0(Q_{2n})(A)$.

First, out of necessity, we prove a quadratic version of Lindel's Theorem ([L], Bass-Quillen Conjecture), on extendibility of projective modules P over polynomial rings $R = A[T]$, where A is a regular ring containing a field, as follows.

Theorem 1.2. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let $R = A[T_1, \ldots, T_n]$ be a polynomial ring. Suppose q is an isotropic quadratic form over k, with $rank(q) = r$. Suppose (P, φ) is a quadratic space over R. Write $(\overline{P}, \overline{\varphi}) := (P, \varphi) \otimes \frac{R}{(T_1, \ldots, T_n)R}$. Assume $(\overline{P}, \overline{\varphi})$ is locally trivial* (in the sense clarified in (3.2)). *Then, (P, φ) is extended from A.*

While Theorem 1.2 is significant by its own right, the theorem completely removes the "infinite field" condition, from the methods in [F1,M2].

For commutative noetherian rings A, we establish (see §5) the existence of a natural involution on $\pi_0(\mathcal{L}O(A, n))$, for all $n \geq 2$, which is a key tool in this article. The possibility of a group structure on the obstruction set $\pi_0(\mathcal{L}O(A, n))$ has always been a part of the Homotopy Program, particularly, in the upper half of the range of n. We establish a group structure on $\pi_0(\mathcal{L}O(A, n))$ as follows.

Theorem 1.3. *Suppose A is a regular ring over a field k, with $1/2 \in k$, with $\dim A = d$. Let $n \geq 2$ be an integer, with $2n \geq d + 2$. Then, there is a structure of an abelian group on $\pi_0(\mathcal{L}O(A, n)) = \pi_0(Q_{2n})(A)$. The addition is determined as follows: Suppose $x = \zeta(K, \omega_K)$, $y = \zeta(I, \omega_I) \in \pi_0(Q_{2n})(A)$, where $\zeta : \mathcal{L}O(A, n) \longrightarrow \pi_0(Q_{2n})(A)$ is the natural map and (K, ω_K), $(I, \omega_I) \in \mathcal{L}O(A, n)$, such that height$(K) \geq n$ and $K + I = A$. Then,*

$$x + y = \zeta(KI, \omega_K \star \omega_I) \in \pi_0(Q_{2n})(A)$$

where $\omega_K \star \omega_I : A^n \twoheadrightarrow \frac{KI}{(KI)^2}$ is the unique surjective map determined by ω_K and ω_I.

In deed, another definition of an Obstruction group $E^d(A)$, where $d = \dim A$, of zero cycles was outlined by Nori, by considering the free abelian group generated by $\{(m, \omega) \in \mathcal{L}O(A, \dim A) : m \in \max(A)\}$, and the relations were obtained using homotopy equivalences (*see, for example, [BS1, §4, pp. 175–176], where it was denoted by $E(A)$*). By suitable adaptations, Bhatwadekar and Sridharan [BS2] defined obstruction groups $E^n(A)$, for each co-dimension $n \geq 0$, where A was assumed to be any noetherian commutative ring (see § A). These groups $E^n(A)$ are known as Euler class groups. Ever since, relationship between the obstructions $\pi_0(\mathcal{L}O(A, n))$ and $E^n(A)$ remained an open question in the Homotopy Program, which is settled follows.

Theorem 1.4. *Suppose A is a regular ring over a field k, with $1/2 \in k$, with $\dim A = d$. Assume n is an integer such that $n \geq 2$ and $2n \geq d+2$. Then, there is a surjective homomorphism, $\rho : E^n(A) \twoheadrightarrow \pi_0(\mathcal{L}O(A,n))$. Further, this homomorphism is an isomorphism, if for orientations $(I, \omega_I) \in \mathcal{L}O(A,n)$, with height$(I) \geq n$, its triviality in $\pi_0(\mathcal{L}O(A,n))$ implies ω_I lifts to a surjective map $A^n \twoheadrightarrow I$.*

In particular, by [BK], ρ is an isomorphisms, if A is essentially smooth and k is an infinite perfect field, and $2n \geq d+3$ (see Theorem 6.15).

Remark 1.5. After this article was posted in arXiv, we became aware of the article [AF], which has some overlap with the results in this article. The results in this article would be an improvement upon those in [AF], and the methods are completely different. Our main results hold for any regular ring A containing a field k, with $1/2 \in k$, while main results in [AF] are valid for smooth rings over infinite perfect fields k, with $1/2 \in k$. Due to uncertainty regarding the validity of [F1, Theorem 3.2.8], (*now retracted* [F2]), there is no valid proof of the injectivity of the map s in Theorem 3.1.9 in [AF], when $2n = d+2$. In this article (Theorem 6.15), we use [BK, Theorem 4.13] to establish the injectivity, when $2n \geq d+3$.

Remark 1.6. Let $\widetilde{CH}^d(A)$ denote the Chow-Witt group, defined by Barge and Morel [BM]. It is known that, under the hypotheses of (1.4), $\pi_0(\mathcal{L}O(A,d)) \cong \pi_0(Q_{2d}(A)) \cong \widetilde{CH}^d(A)$, where the first isomorphism is established in Lemma 2.3 and the second isomorphism was studied in [AF2].

It follows from Theorem 1.4 that, under the same hypotheses, there is an isomorphism $E^d(A) \xrightarrow{\sim} \widetilde{CH}^d(A)$. This settles a long standing open question (see [Mo, Remark 1.33(2)]), as was also claimed in [AF, Theorem 3 (3)]. However, the proof of [AF, Theorem 3 (3)] remains incomplete, as clarified above (1.5).

The organization of this article is fairly sequential, as the results are described above. We add that, in Section 4, some important homotopy theorems were established. This, in particular, establishes that homotopy relations described above, is actually an equivalence relation, when the ring A is regular over a field k, with $1/2 \in$. In Section A, we give some preliminaries about Euler class groups. In Section B, we provide some background regarding Elementary Orthogonal Subgroups $\mathscr{E}O(A, q_{2n+1})$.

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2. Notations and preliminaries

First, we establish some notations, some of which may be standard.

Notations 2.1. Throughout, k will denote a field (or a ring), with $1/2 \in k$ and A will denote a commutative noetherian rings. Denote $\tilde{q}_{2n+1} = \sum_{i=1}^n X_i Y_i + Z(Z-1)$,

$$\mathscr{A}_{2n} = \frac{k[X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z]}{(\tilde{q}_{2n+1})} \quad \text{and} \quad Q_{2n} = \operatorname{Spec}(\mathscr{A}_{2n}). \tag{1}$$

Accordingly, for a commutative ring A, denote

$$Q_{2n}(A) = \left\{ (s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in A^{2n+1} : \sum_{i=1}^n f_i g_i + s(s-1) = 0 \right\}$$

For $\mathbf{v} = (s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in Q_{2n}(A)$, denote the ideals

$$\begin{cases} \mathbb{I}(\mathbf{v}) := (f_1, \ldots, f_n, s)A \\ \mathbb{J}(\mathbf{v}) := (f_1, \ldots, f_n, 1-s)A \end{cases} \quad \text{and surjective maps} \quad \begin{cases} \omega_{\mathbf{v}} : A^n \to \frac{\mathbb{I}(\mathbf{v})}{\mathbb{I}(\mathbf{v})^2} \\ \omega'_{\mathbf{v}} : A^n \to \frac{\mathbb{J}(\mathbf{v})}{\mathbb{J}(\mathbf{v})^2} \end{cases}$$

Also, $\underline{Sets}$ will denote the category of sets. The homotopy pre-sheaves are given by the pushout diagrams in $\underline{Sets}$:

$$\begin{array}{ccc} Q_{2n}(A[T]) & \xrightarrow{T=0} & Q_{2n}(A) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ Q_{2n}(A) & \longrightarrow & \pi_0(Q_{2n})(A) \end{array} \tag{2}$$

Further, consider the quadratic form $q_{2n+1} = \sum_{i=1}^{n} X_i Y_i + Z^2$, and denote

$$\mathscr{B}_{2n} = \frac{k[X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z]}{(q_{2n+1} - 1)}, \quad Q'_{2n} = \mathrm{Spec}\,(\mathscr{B}_{2n})\,. \tag{3}$$

Accordingly,

$$Q'_{2n}(A) = \left\{ (s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in A^{2n+1} : \sum_{i=1}^{n} f_i g_i + s^2 - 1 = 0 \right\}$$

Since, $1/2 \in k$ there is an isomorphism $\alpha : \mathscr{A}_{2n} \xrightarrow{\sim} \mathscr{B}_{2n}$, which induces bijective correspondence

$$\alpha : Q'_{2n}(A) \xrightarrow{\sim} Q_{2n}(A) \quad \text{and its inverse} \quad \beta : Q_{2n}(A) \xrightarrow{\sim} Q'_{2n}(A)$$

Now, the action of the Orthogonal groups $O(A, q_{2n+1})$ on $Q'_{2n}(A)$ translates to an action of $O(A, q_{2n+1})$ on $Q_{2n}(A)$ as follows:

$$\forall\, \mathbf{v} \in Q_{2n}(A), M \in O(A, q_{2n+1}) \quad \text{define} \quad \mathbf{v} * M := \alpha(\beta(\mathbf{v})M)$$

The local orientations of an ideal are defined as follows.



Definition 2.2. *Suppose A is a commutative ring and I is an ideal in A. For an integer $n \geq 2$, a* ***local n-orientation*** *of I is a pair (I, ω), where $\omega : A^n \twoheadrightarrow I/I^2$ is a surjective homomorphism. Such a local n-orientation is determined by any set of elements $f_1, \ldots, f_n \in I$ such that $I = (f_1, \ldots, f_n) + I^2$. Given such a set of generators $f_1, \ldots, f_n$ of I/I^2, there is an element $s \in I$ and $g_1, \ldots, g_n$ such that $\sum_{i=1}^{n} f_i g_i + s(s-1) = 0$. Note,*

$$(s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in Q_{2n}(A).$$

$$\textit{Write} \quad \zeta(I, \omega) := [(s; f_1, \ldots, f_n; g_1, \ldots, g_n)] \in \pi_0(Q_{2n}(A))$$

This association is well defined ([F1, Theorem 2.0.7]). We refer to $\zeta(I, \omega)$, as the ***homotopy obstruction class****. The set of all n-orientations (I, ω_I) will be denoted by $\mathcal{L}O(A, n)$. Therefore, we have a commutative diagram*

$$\begin{array}{ccc} Q_{2n}(A) & & \\ {\scriptstyle \eta}\downarrow & \searrow^{\zeta_0} & \\ \mathcal{L}O(A, n) & \xrightarrow[\zeta]{} & \pi_0(Q_{2n}(A)) \end{array} \qquad \text{where} \quad \eta(\mathbf{v}) = (\mathbb{I}(\mathbf{v}), \omega_{\mathbf{v}}). \tag{4}$$

Analogous to the definition of $\pi_0(Q_{2n})(A)$, we define $\pi_0(\mathcal{LO}(A, n))$, by the pushout diagram

$$\begin{array}{ccc} \mathcal{L}O(A[T], n) & \xrightarrow{T=0} & \mathcal{L}O(A, n) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ \mathcal{L}O(A, n) & \longrightarrow & \pi_0(\mathcal{L}O(A, n)) \end{array} \qquad \text{in} \quad \underline{Sets}. \tag{5}$$

The above homotopy obstruction set $\pi_0(\mathcal{L}O(A, n))$ was among the ideas envisioned by Nori (around 1990). Following lemma establishes that $\pi_0(Q_{2n})(A)$ is in bijection with $\pi_0(\mathcal{L}O(A, n))$.

Lemma 2.3. *Suppose A is a commutative noetherian ring, with* $\dim A = d$ *and $n \geq 2$ is an integer. Then, the map $\zeta : \mathcal{L}O(A, n) \longrightarrow \pi_0(Q_{2n})(A)$ induces bijective map*

$$\overline{\zeta} : \pi_0(\mathcal{L}O(A, n)) \xrightarrow{\sim} \pi_0(Q_{2n})(A).$$

The inverse map is induced by η.

Proof. Given a $\tilde{H} \in \pi_0(\mathcal{L}O(A[T], n))$, $\eta(H(T)) = \tilde{H}$ for some $H(T) \in Q_{2n}(A[T])$. So, $\zeta(\tilde{H}(0)) = \zeta_0(H(0)) = \zeta_0(H(1)) = \zeta(\tilde{H}(1))$. This establishes that ζ factors through a set theoretic map $\overline{\zeta} : \pi_0(\mathcal{L}O(A, n)) \longrightarrow \pi_0(Q_{2n})(A)$. Since η is on to , so is $\overline{\zeta}$. There is also a well defined map $\overline{\eta} : \pi_0(Q_{2n})(A) \longrightarrow \pi_0(\mathcal{L}O(A, n))$ induced by η. It is clear that $\overline{\eta}\overline{\zeta} = 1$. So, $\overline{\zeta}$ is also one to one and hence is bijective. The proof is complete. □

The following "moving lemma argument" is fairly standard. A number of variations of the same (2.4) would be among the frequently used tools for the rest of our discussions.

Lemma 2.4 (Moving Lemma). *Suppose A is a commutative noetherian ring with* $\dim A = d$ *and $n \geq 2$ is an integer such that $2n \geq d + 1$. Let $K \subseteq A$ be an ideal with height$(K) \geq n$ and $(I, \omega_I) \in \mathcal{LO}(A, n)$. Then, there is an element $\mathbf{v} = (s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in Q_{2n}(A)$ such that $\eta(\mathbf{v}) = (I, \omega_I)$. Further, with $J = \mathbb{J}(\mathbf{v})$, we have height$(J) \geq n$ and $J + K = A$.*

Proof. We use the standard basis $e_1, \ldots, e_n$ of A^n. Let $a_1, \ldots, a_n \in I$ be such that $\omega_I(e_i) = a_i + I^2$. So, $I = (a_1, \ldots, a_n) + I^2$. Using Nakayama's Lemma, there is an element $t \in I$, such that $t(1 - t) = \sum_{i=0}^{n} a_i b_i$ for some $b_1, \ldots, b_n \in A$ and $I = (a_1, \ldots, a_n, t)$. (*Readers are referred to [M1] regarding generalities on Basic Element Theory and generalized dimension functions*.) Write

$$\mathscr{P} = \{\wp \in \text{Spec}\,(A) : t \notin \wp, \text{ and either } K \subseteq \wp \text{ or } \textit{height}\,(\wp) \leq n - 1\}$$

There is a generalized dimension function (see [M1]) $\delta : \mathscr{P} \longrightarrow \mathbb{N}$, such that $\delta(\wp) \leq n - 1 \; \forall \; \wp \in \mathscr{P}$. Now $(a_1, \ldots, a_n, t^2) \in A^{n+1}$ is basic on $\mathscr{P}$. So, there are $\lambda_1, \ldots, \lambda_n \in A$ such that $(a_1 + \lambda_1 t^2, \ldots, a_n + \lambda_n t^2) \in A^n$ is basic on $\mathscr{P}$. For $i = 1, \ldots, n$, denote $f_i := a_n + \lambda_i t^2$. Then, $\omega_I(e_i) = f_i + I^2$ and hence $I = (f_1, \ldots, f_n) + I^2$. Using Nakayama's Lemma, there is an element $s \in I$ such that $(1 - s)J \subseteq (f_1, \ldots, f_n)$. Hence, $s(1 - s) = \sum_{i=1}^{n} f_i g_i$ for some $g_1, \ldots, g_n$. Now, $\mathbf{v} = (s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in Q_{2n}(A)$ has the desired properties. The proof is complete. □

Remark 2.5. In Moving Lemma 2.4, we proved that, given

$$\mathbf{u} = (t; a_1, \ldots, a_n; b_1, \ldots, b_n) \in Q_{2n}(A), \text{ there is}$$
$$\mathbf{v} = (s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in Q_{2n}(A), \text{ such that}$$

1. $\forall\, i = 1, \ldots, n \; f_i = a_i + \lambda_i t^2$,
2. $\eta(\mathbf{u}) = \eta(\mathbf{v})$,
3. *height*$(\mathbb{J}(\mathbf{v})) \geq n$ and $\mathbb{J}(\mathbf{v}) + K = A$.

For the convenience of our discussions, we include some auxiliary notations.

Notations 2.6. Throughout, A will denote a noetherian commutative ring, with $\dim A = d$.

1. For an A-module M and an ideal I of A, homomorphisms $f : M \longrightarrow \frac{I}{I^2}$ would be identified with the induced maps $\frac{M}{IM} \longrightarrow \frac{I}{I^2}$.
2. Let I_1, I_2 be two ideals, with $I_1 + I_2 = A$. For an integer $n \geq 2$, for $i = 1, 2$ let $\omega_i : A^n \twoheadrightarrow \frac{I_i}{I_i^2}$ be two surjective maps. Then, $\omega_1 \star \omega_2 : A^n \twoheadrightarrow \frac{I_1 I_2}{(I_1 I_2)^2}$ will denote the unique surjective map induced by ω_1, ω_2.
3. For integers $n \geq 2$, denote $\mathbf{0} := (0; 0, \ldots, 0; 0, \ldots, 0) \in Q_{2n}(A)$ and $\mathbf{1} := (1; 0, \ldots 0; 0, \ldots, 0) \in Q_{2n}(A)$. Either one of them could be a candidate for the base point of $Q_{2n}(A)$.

4. An element $H(T) \in Q_{2n}(A[T])$ or $H(T) \in Q'_{2n}(A[T])$ would be referred to as a homotopy.
5. We caution the readers that notations in this section would be part of our standard notations, throughout this article. In particular, that would include $\mathbb{I}(\mathbf{v})$, $\mathbb{J}(\mathbf{v})$, $\omega_{\mathbf{v}}$, η, ζ_0, ζ and others.

3. A quadratic version of Lindel's theorem

For an essentially smooth ring A over a field k, and a polynomial ring $R = A[T_1, \dots, T_n]$, Lindel [L] settled Bass-Quillen Conjecture, by proving that finitely generated projective R-modules are extended from A. By the Desingularization Theorem of Popescu ([P], [Sw1, Corollary 1.2]), it follows that the same is true, when A is any regular ring containing a field k (see [Sw1, Theorem 2.1]). In this section, we give a version of the same for Quadratic spaces.

For the convenience of the readers, we recall the following definition.

Definition 3.1. *Suppose A is a noetherian commutative ring, with $1/2 \in A$. Then, a Quadratic space on A is a pair (P, φ), where P is a finitely generated projective A-module and $\varphi : P \xrightarrow{\sim} P^*$ is an isomorphism, such that $\varphi^* = \varphi$. Maps between two quadratic spaces over A, are called orthogonal maps and such isomorphisms are called isometries.*

First, we prove the Quadratic analogue of Lindel's theorem for smooth rings A over perfect fields, as follows.

Proposition 3.2. *Suppose A is an essentially smooth ring over a prefect field k, with $1/2 \in k$ and $\dim A = d$. Let $R = A[T_1, \dots, T_n]$ be a polynomial ring. Suppose q is an isotropic quadratic form over k, with $rank(q) = r$. Suppose (P, φ) is a quadratic space over R. Write $(\overline{P}, \overline{\varphi}) := (P, \varphi) \otimes \frac{R}{(T_1, \dots, T_n)R}$. Assume $(\overline{P}, \overline{\varphi})$ is locally isometric to (A^n, q); meaning $\forall\, \wp \in \mathrm{Spec}\,(A)$, $(\overline{P}, \overline{\varphi})_\wp \cong (A^n, q)_\wp$ are isometric.* (This hypothesis is referred to as "local triviality property"). *Then, (P, φ) is extended from A.*

Proof. We prove by induction on $\dim A = d$. Suppose $d = \dim A = 0$. Then, $A = k$ is a field. By hypothesis $WittIndex(\overline{P}) \geq 1$. So, the theorem is valid, by the Theorem of Ojanguren [O1] (also see [K, pp.425, Thm 6.2.6]). So, we assume $\dim A \geq 1$. We can also assume that A is local (see [K, pp. 419, Thm 5.3.4]). By hypothesis on local triviality of $(\overline{P}, \overline{\varphi})$, there is an isometry $\boldsymbol{\sigma_0 : (A^r, q) \xrightarrow{\sim} (\overline{P}, \overline{\varphi})}$.

By Theorem of Lindel [L] (see [M1, 7.1.1]), there is local subring $B \subseteq A$ such that

1. $B = k[X_1, \dots, X_d]_M$ where $X_1, \dots, X_n$ are variables, $f(X_1) \in k[X_1]$ and $M = (f(X_1), X_2, \dots, X_d)$ is a maximal ideal.
2. There is an element $h \in MB$ such that the inclusion map $B \hookrightarrow A$ is an **analytic isomorphism**; meaning

$$\begin{cases} A = B + Ah \\ \forall\, n \in \mathbb{N} \quad Bh^n = B \cap Ah^n. \end{cases} \qquad \text{Consequently,} \quad \begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B_h & \longrightarrow & A_h \end{array} \tag{6}$$

is a patching diagram (see [O2,R] regarding such patching diagrams).

We denote

$$F' = B^n,\ F'[\underline{T}] = F' \otimes B[T_1, \dots, T_n],\ F = A^n,\ F[\underline{T}] = F \otimes A[T_1, \dots, T_n].$$

So, $\boldsymbol{\sigma_0 : (F, q) \xrightarrow{\sim} (\overline{P}, \overline{\varphi})}$, above, is an isometry. Since, $\dim A_h \leq d - 1$, by induction, there is a quadratic space (Q, ψ) on A_h, and an isometry

$$\sigma_1 : (Q, \psi) \otimes A_h[T_1, \dots, T_n] \xrightarrow{\sim} (P_h, \varphi_h).$$

Let "overline" denote modulo $(T_1, \dots, T_n)$. Then σ induces an isometry,

$$\overline{\sigma}_1 : (Q, \psi) \xrightarrow{\sim} (\overline{P}_h, \overline{\varphi}_h). \quad \text{So, } \tau := \bar{\sigma}_1^{-1}(\sigma_0)_h : (F_h, q) \xrightarrow{\sim} (Q, \psi) \text{ is an isometry.}$$

Further, consider the obvious isometry $\sigma_2 : (F' \otimes A_h, q) \xrightarrow{\sim} (F_h, q)$. Combining, all these, we have commutative diagram of isometries

$$\begin{array}{ccccc} (F' \otimes A_h[\underline{T}], q) & \xrightarrow{\sigma_2 \otimes 1} & (F_h[\underline{T}], q) & \xrightarrow{\tau \otimes 1} & (Q[\underline{T}], \psi \otimes 1) \\ & \searrow^{\sigma} & & & \downarrow \sigma_1 \\ & & & & P_h \end{array}$$

where σ is defined by composition. Notice $\sigma \otimes \frac{A_h[\underline{T}]}{(\underline{T})} = (\sigma_0)_h \sigma_2$. With respect to the patching diagram (6), consider the patching diagram (see [O2,R])

$$\begin{array}{ccccc} (\widetilde{P}, \widetilde{\varphi}) & & \longrightarrow & & (P, \varphi) \\ \downarrow & & & & \downarrow \\ (F'_h[\underline{T}], q) & \longrightarrow & (F' \otimes A_h[\underline{T}], q) & \xrightarrow[\sigma]{} & (P, \varphi)_h \end{array}$$

Here $\widetilde{P}$ is obtained by this patching $F'_h[\underline{T}]$ and P, via σ. Then, $\widetilde{P}$ has a structure of a quadratic space $(\widetilde{P}, \widetilde{\varphi})$ [O2, Theorem 8], making the above a patching diagram of quadratic spaces. Since, $\sigma \otimes \frac{A_h[\underline{T}]}{(\underline{T})} = (\sigma_0)_h \sigma_2$, it follows that, there is an isometry $(F', q) \xrightarrow{\sim} (\widetilde{P}, \widetilde{\varphi}) \otimes \frac{B[\underline{T}]}{(\underline{T})}$. (i.e. local triviality property is preserved). Now, replacing A by B and P by $\widetilde{P}$, we can assume $A = k[X_1, \ldots, X_d]_M$ with $M = (f(X_1), X_2, \ldots, X_d)$.

Write $A_0 = k[X_1, \ldots, X_{d-1}]_{\mathfrak{m}}$, where $\mathfrak{m} = (f(X_1), X_2, \ldots, X_{d-1})$. Then, $A_0[X_d] \hookrightarrow A$ is an **analytic isomorphism** along X_d. Now, the new patching diagram looks like

$$\begin{array}{ccc} A_0[X_d] & \longrightarrow & A \\ \downarrow & & \downarrow \\ A_0[X_d, X_d^{-1}] & \longrightarrow & A_{X_d} \end{array}$$

Repeating the same method,

$$(P, \varphi) \cong (\widetilde{P}, \widetilde{\varphi}) \otimes_{A_0[X_d, T_1, \ldots, T_n]} A[T_1, \ldots, T_n]$$

where $(\widetilde{P}, \widetilde{\varphi})$ is a quadratic space over $A_0[X_d, T_1, \ldots, T_n]$. Since $\dim A_0 = d - 1$, by induction, $(\widetilde{P}, \widetilde{\varphi})$ is extended from A_0, and hence from $A_0[X_d]$. Therefore, (P, φ) is extended from A. This completes the proof. □

Remark 3.3. Our interest in this question of extendibility of Quadratic spaces over polynomial rings was triggered, by the reference to [AHW, Theorem 3.3.7] in the proof of [F1, Theorem 1.0.6]. It was confusing that a natural quadratic analogue of Lindel's theorem [L] would not be considered in [F1], instead, if such was available in the literature. The statement of [AHW, Theorem 3.3.6] obscures the simplicity of such an analogue. However, the referee points to the references [Pr1,Pr2] and most importanly [Rr, Theorem 1.3]. The result of Ravi Rao [Rr, Proposition 1.3] is, perhaps, the most significant result on this, available in the literature, which can be viewed as a local case of Proposition 3.2, above. Proofs of both [Rr, Proposition 1.3] and Proposition 3.2 would be similar to the proof of Lindel's theorem [L]. We further strengthen Proposition 3.2, as follows (3.5), by using Popescu's Desingularization Theorem [P].

For the convenience of our discussions, we state the following lemma, which can be checked locally.

Lemma 3.4. *Let R be a noetherian commutative ring and $A \subseteq R$ be a noetherian subring. Let P, Q be two finitely generated projective A-modules. Then, the map $Hom_A(P, Q) \longrightarrow Hom_R(P \otimes R, Q \otimes R)$ is injective.*

Now, we use Popescu's Desingularization Theorem ([P], [Sw1, Corollary 1.2]), to remove the perfectness condition in (3.2), as follows (3.5).

Theorem 3.5. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let $R = A[T_1, \dots, T_n]$ be a polynomial ring. Suppose q is an isotropic quadratic form over k, with $rank(q) = r$. Suppose (P, φ) is a quadratic space over R. Write $(\overline{P}, \overline{\varphi}) := (P, \varphi) \otimes \frac{R}{(T_1,\dots,T_n)R}$. Assume $(\overline{P}, \overline{\varphi})$ is locally trivial* (in the sense clarified in (3.2)). *Then, (P, φ) is extended from A.*

Proof. For any matrix M, denote the i^{th}-column of M by $\boldsymbol{M^{(i)}}$. First, $P \subseteq R^N$, is image of an idempotent matrix $\iota : R^N \longrightarrow R^N$. In particular, P is generated by the columns $\iota^{(i)}$ of ι. Denote $Q = \ker(\iota)$. We display two exact sequences and commutative diagrams of maps:

$$0 \longrightarrow Q \longrightarrow R^N \longrightarrow P \longrightarrow 0, \qquad 0 \longrightarrow Q^* \longrightarrow (R^N)^* \longrightarrow P^* \longrightarrow 0$$

$$\text{Write} \quad R^N = \oplus_{i=1}^N Re_i, \quad (R^N)^* = \oplus_{i=1}^N Re_i^*.$$

The latter diagram shows that P^* is generated by the columns of ι^*. The quadratic structure on P is given by an isomorphism $\varphi : P \longrightarrow P^*$. For $\sigma \in End(P, P^*)$, we extend $\sigma : R^N \longrightarrow (R^N)^*$, by defining $\sigma_{|Q} = 0$. So, we have

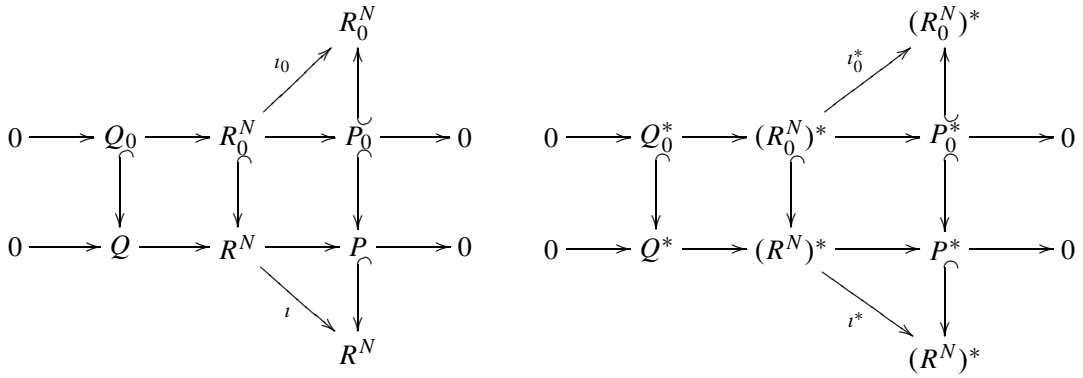
$$End(P, P^*) \subseteq End(R^N, (R^N)^*) = \mathbb{M}_N(R).$$

So, σ is given by a matrix $\Sigma \in \mathbb{M}_N(R)$. Then, Σ has the following properties: (1) $\Sigma^t = \Sigma$, (2) $\Sigma q = 0$ for all $q \in Q$. (3) Σ is injective on P, (4) $image(\Sigma) = P^*$.

Let $\varepsilon_i := (\iota^*)^{(i)}$ the columns of ι^*. Then, $\{\varepsilon_i\}$ generates P^*. Now,

$$\exists\ p_i \in P\ \ni\ \varepsilon_i = \Sigma(p_i). \quad \text{Write} \quad p_i = \sum_j \lambda_{ij}(T)\iota^{(i)} \tag{7}$$

Now, let S be the set of all coefficients of entires in ι, Σ, q and of λ_{ij}. Let $\mathbb{F}$ be the prime field of k. Let $A_0 = \mathbb{F}[S] \subseteq A$ and $R_0 = A_0[T_1, \dots, T_n]$. Let $\iota_0 : R_0^N \longrightarrow R_0^N$ be given by the matrix of ι. Let $P_0 = image(\iota_0)$, $Q_0 = \ker(\iota_0)$. Also, consider $\Sigma_0 = \Sigma : R_0^n \longrightarrow (R_0^N)^*$. We have the commutative diagrams



Now, Then,

1. From equation (7), $\varepsilon_i \in P_0^*$ and $p_i \in P_0$.
2. $\Sigma_0^t = \Sigma^t = \Sigma = \Sigma_0$.
3. $\Sigma_0 q = 0$ for all $q \in Q_0$. This is because $Q_0 \subseteq Q$.
4. Σ is injective on P_0. This is because $P_0 \subseteq P$.
5. $image(\Sigma_0) = P_0^*$.

To see this, note from the diagram $Q_0 = Q \cap R_0^N$. Likewise $P \cap R_0^N = P_0$ and $P^* \cap (R_0^N)^* = P_0^*$. So,

$$\Sigma_0(R_0^N) \subseteq \Sigma(R^N) \cap (R_0^N)^* \subseteq P^* \cap (R_0^N)^* = P_0^*$$

Note, P_0^* is generated by the columns of ι^* (i. e. ε_i), as an R_0-module. By equation (7), since $p_i \in P_0$, we have Σ_0 maps on to P_0^*.

So, Σ_0 defines an isomorphism $\varphi_0 : P_0 \xrightarrow{\sim} P_0^*$, such that $\varphi_0^* = \varphi_0$, because $\Sigma = \Sigma_0$ is symmetric. Also, $\varphi_0 \otimes R = \varphi$, because they are restriction of the maps defined by $\Sigma_0 = \Sigma$. Therefore, $\boldsymbol{(P_0, \varphi_0) \otimes R = (P, \varphi)}$.

Now, we will enlarge A_0 to accommodate the local triviality condition. There are $s_1, \ldots, s_m \in A$ and isometries $\psi_i : (A_{s_i}, q) \xrightarrow{\sim} (\overline{P}_{s_i}, \varphi_{s_i})$ such that $s_1 + \cdots + s_m = 1$. Let $e_1, \ldots, e_r$ be the standard basis of A^r. Fix l and work with $\psi_l : (A^r_{s_l}, q) \xrightarrow{\sim} (\overline{P}_{s_l}, \varphi_{s_l})$. The generators of $\overline{P}$ is given by the columns $\overline{\iota^{(j)}}$. So,

$$\psi_l(e_i) = \sum_j \frac{a_{ijl}}{s_1^\mu} \overline{\iota^{(j)}}. \quad \text{Also,} \quad \overline{\iota^{(i)}} = \sum_j \frac{b_{ijl}}{s_1^\mu} \psi_l(e_j)$$

Write $A_1 = A_0[s_i, a_{ijl}, b_{ijl}]$ and replace A_0 by A_1. Then, $\psi_l(e_i) \in \overline{P}_{0s_l}$. This defines a map $\eta_l : (A_0^l)_{s_i} \longrightarrow \overline{P}_{0s_l}$, by setting $\boldsymbol{\eta_l(e_i) = \psi_l(e_i)}$. Then, $\eta_l \otimes 1 = \psi_l$. By the latter equation on $\overline{\iota^{(i)}}$, it follows η_i is surjective, and hence an isomorphism. To prove that η_i is isometry, we need to check the commutativity of the diagrams

$$\begin{array}{ccc} ((A_0^r)_{s_l}, q) & \xrightarrow{\eta_l} & (\overline{P}_0, \varphi_0)_{s_l} \\ \downarrow & & \downarrow \\ ((A_0^r)_{s_l}, q)^* & \xleftarrow[\eta_l^*]{} & (\overline{P}_0, \varphi_0)^*_{s_l} \end{array}$$

The commutativity follows from Lemma 3.4. This establishes that (P_0, φ_0) is locally trivial.

By the Theorem of Popescu [P], we have the following commutative the diagram of rings and homomorphisms

$$\begin{array}{ccccc} & & & & A_2 \\ & & & \overset{\beta}{\nearrow} & \downarrow \\ \mathbb{F} & \longrightarrow & A_0 & \hookrightarrow & A \end{array} \qquad \text{where } A_2 \text{ is smooth over } \mathbb{F}.$$

Write $R_2 = A_2[T_1, \ldots, T_n]$. Then, by (3.2), $(P_0, \varphi_0) \otimes R_2$ is extended from A_2. Since $(P, \varphi) = ((P_0, \varphi_0) \otimes R_2) \otimes R$, we have (P, φ) is extended from A. This completes the proof. □

4. Homotopy

In this section, we use the Quadratic analogue (3.5) of Lindel's theorem to prove some key homotopy theorems. First, we recall the following standard lemma.

Lemma 4.1. *Suppose $(A, \mathfrak{m})$ is a commutative noetherian local ring and $\mathbf{u} \in Q'_{2n}(A)$. Consider the orthogonal complement $K = A\mathbf{u}^\perp \subseteq (A^{2n+1}, q_{2n+1})$. Then, $K \cong (A, q_{2n})$, where $q_{2n} = \sum_{i=1}^n X_i Y_i$.*

Proof. Write $\mathbf{u}_0 = (1; 0, \ldots, 0; 0, \ldots, 0) \in Q'_{2n}(A)$. By Lemma B.3, there is a matrix $\sigma \in \mathscr{E}O(A, q_{2n+1})$, such that $\mathbf{u}_0\sigma = \mathbf{u}$. So, we have a diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & A^{2n+1} & \xrightarrow{\langle \mathbf{u}, - \rangle} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_0} & & \downarrow{\scriptstyle \sigma} & & \| & & \\ 0 & \longrightarrow & (A, q_{2n}) & \longrightarrow & A^{2n+1} & \xrightarrow[\langle \mathbf{u}_0, - \rangle]{} & A & \longrightarrow & 0 \end{array}$$

So, σ induces an isometry $\sigma_0 : K \xrightarrow{\sim} (A, q_{2n})$. The proof is complete. □

Theorem 4.2. *Let A be a regular ring over a field k, with $1/2 \in k$. Suppose $H(T) \in Q'_{2n}(A[T])$. Then, there is an orthogonal matrix $\sigma(T) \in O(A[T], q_{2n+1})$, such that*

$$H(T) = H(0)\sigma(T) \quad \text{and} \quad \sigma(0) = 1.$$

Proof. Suppose $H(T) \in Q'_{2n}(A[T])$ is a homotopy. For $R = A[T]$, A, use the following generic notations, to denote the quadratic modules

$$\begin{cases} q := q_{2n+1} : R^{2n+1} \to R & \text{sending} \quad (u_1, \dots, u_n, v_1, \dots, v_n, s) \mapsto \sum_{i=1}^n u_i v_i + s^2 \\ q_0 : R \to R & \text{sending} \quad s \mapsto s^2 \end{cases}$$

As usual, define $B_q(e, e') = \frac{q(e+e')-q(e)-q(e')}{2}$. With respect to the standard basis, the matrix of B_q is given by

$$B_q := \frac{1}{2}\begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

So, the map $R^{2n+1} \to (R^{2n+1})^*$ sends $\mathbf{v} \mapsto \mathbf{v}B_q$. These bilinear forms give the following exact sequences:

$$0 \longrightarrow K \longrightarrow A[T]^{2n+1} \xrightarrow{\langle H(T), - \rangle} A[T] \longrightarrow 0$$

$$0 \longrightarrow K_0 \longrightarrow A^{2n+1} \xrightarrow[\langle H(0), - \rangle]{} A \longrightarrow 0$$

Therefore, $K = (A[T]H(T))^{\perp}$, $K_0 = (AH(0))^{\perp}$ are orthogonal complements. Write $\overline{K} := K \otimes \frac{A[T]}{(T)}$. By Lemma 4.1, $\overline{K}$ is locally isometric to (A, q_{2n}). By Theorem 3.5, there is an isometry $\tau : K \xrightarrow{\sim} \overline{K} \otimes A[T]$. Further, it follows $\overline{K} = (RH(0))^{\perp} \cong K_0$. Therefore, there is an isometry $\sigma_0 : \overline{K} \xrightarrow{\sim} K_0$, which extends to an isometry $\sigma_0 \otimes 1 : \overline{K} \otimes A[T] \xrightarrow{\sim} K_0 \otimes A[T]$. Then, $\sigma_1 := (\sigma_0 \otimes 1)\tau : K \xrightarrow{\sim} K_0 \otimes A[T]$ is an isometry. Finally, note

$$(A[T]H(T), q_{|A[T]H(T)}) \cong (A[T], q_0) \cong (A[T]H(0), q_{|A[T]H(0)}).$$

Now, consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & A[T]^{2n+1} & \xrightarrow{\langle H(T), - \rangle} & A[T] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma(T)} & & \| & & \\ 0 & \longrightarrow & K_0 \otimes A[T] & \longrightarrow & A[T]^{2n+1} & \xrightarrow[\langle H(0), - \rangle]{} & A[T] & \longrightarrow & 0 \end{array} \tag{8}$$

of quadratic spaces. In this diagram, the horizontal lines are split exact sequences of quadratic spaces. Hence, there is an isometry $\sigma(T) \in O(A[T], q)$, such that the diagram (8) commutes. Therefore $H(0)\sigma(T) = H(T)$. By construction, (*alternately, by replacing* $\sigma(T)$ by $\sigma(0)^{-1}\sigma(T)$), we have $\sigma(0) = 1$. The proof is complete. □

The following Corollary would be of some use for our future discussions.

Corollary 4.3. *Let A be a regular ring over a field k, with $1/2 \in k$ and $n \geq 2$ be an integer. Let $\mathbf{u}, \mathbf{v} \in Q'_{2n}(A)$ such that $[\mathbf{u}] = [\mathbf{v}] \in \pi_0(Q'_{2n})(A)$. Then, there is a homotopy $H(T) \in Q'_{2n}(A[T])$ such that $H(0) = \mathbf{u}$ and $H(1) = \mathbf{v}$. Equivalently, for $\mathbf{u}, \mathbf{v} \in Q_{2n}(A)$ if $\zeta_0(\mathbf{u}) = \zeta_0(\mathbf{v}) \in \pi_0(Q_{2n})(A)$, then there is a homotopy $H(T) \in Q_{2n}(A[T])$ such that $H(0) = \mathbf{u}$ and $H(1) = \mathbf{v}$.*

Proof. Suppose $\mathbf{u}, \mathbf{v} \in Q'_{2n}(A)$ such that $[\mathbf{u}] = [\mathbf{v}] \in \pi_0(Q'_{2n})(A)$. Then, there is a sequence of homotopies $H_1(T), \dots, H_m(T) \in Q'_{2n}(A[T], q_{2n+1})$ such that $\mathbf{u}_0 := H_1(0) = \mathbf{u}$, $\mathbf{u}_m := H_m(1) = \mathbf{v}$ and $\forall\ i = 1, \dots, m-1$, we have $\mathbf{u}_i := H_i(1) = H_{i+1}(0)$. By Theorem 4.2, for $i = 1, \dots, m$ there are orthogonal matrices $\sigma_i(T) \in O(A[T], q_{2n+1})$ such that $\sigma_i(0) = 1$ and $H_i(T) = H_i(0)\sigma_i(T) = \mathbf{u}_{i-1}\sigma_i(T)$. Therefore, $\mathbf{u}_i = H_i(1) = \mathbf{u}_{i-1}\sigma_i(1)$. Write $H(T) = \mathbf{u}_0\sigma_1(T) \cdots \sigma_m(T)$. Then, $H(T) \in Q'_{2n}(A[T])$ and $H(0) = \mathbf{u}_0$ and $H(1) = \mathbf{u}_m$. This establishes first part of the statement on $\pi_0(Q'_{2n})(A)$. The latter assertion on $\pi_0(Q_{2n})(A)$ follows from the former, by the bijective correspondences $Q'_{2n}(A) \longleftrightarrow Q_{2n}(A)$ and $Q'_{2n}(A[T]) \longleftrightarrow Q_{2n}(A[T])$. This completes the proof. □

Remark 4.4. Another way to state (4.3) would be to say that the homotopy relation on $Q_{2n}(A)$ is actually an equivalence relation.

In a slightly more formal language, the above is summarized as follows.

Theorem 4.5. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $n \geq 2$ is an integer. For, $\sigma(T) \in O(A[T], q_{2n+1})$ and $\mathbf{u} \in Q'_{2n}(A)$, define the* (right) *action $\mathbf{u}\sigma(T) := \mathbf{u}\sigma(1) \in Q'_{2n}(A)$. Denote $O(A, q_{2n+1}, T) = \{\sigma(T) \in O(A[T], q_{2n+1}) : \sigma(0) = 1\}$. Then, the map*

$$\frac{Q'_{2n}(A)}{O(A, q_{2n+1}, T)} \longrightarrow \pi_0(Q_{2n})(A) \quad \text{is a bijection.}$$

Proof. Similar to the proof of (4.3). □

Remark 4.6. A version of Theorem 4.5 is embedded in the proof of [F1, Theorem 1.0.6], where it was assumed that A essentially smooth over an infinite field k, with $1/2 \in k$. In [F1], methods of Nisnevich topology was used [AHW], while our methods are fairly basic.

5. The involution

To begin with, we introduce the following key definition, in this article.

Definition 5.1. *Suppose A is a commutative ring. For $\mathbf{v} = (s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in Q_{2n}(A)$ define*

$$\Gamma(s; f_1, \ldots, f_n; g_1, \ldots, g_n) = (1 - s; f_1, \ldots, f_n; g_1, \ldots, g_n)$$

This association, $\mathbf{v} \mapsto \Gamma(\mathbf{v})$, establishes a bijective correspondence

$$\Gamma : Q_{2n}(A) \xrightarrow{\sim} Q_{2n}(A), \quad \text{such that} \quad \Gamma^2 = 1_{Q_{2n}(A)}.$$

That means, Γ is an involution on $Q_{2n}(A)$. (This notation Γ will be among the standard notations throughout this article.)

We record the following obvious lemma.

Lemma 5.2. *Suppose A is a commutative noetherian ring and $n \geq 2$ is an integer. Let $\Gamma : Q_{2n}(A) \xrightarrow{\sim} Q_{2n}(A)$ be the involution, as in (5.1) and $\mathbf{v} = (s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in Q_{2n}(A)$. Then,*

1. $\mathbb{I}(\mathbf{v}) \cap \mathbb{J}(\mathbf{v}) = (f_1, f_2, \ldots, f_n)$.
2. $\mathbb{J}(\mathbf{v}) = \mathbb{I}(\Gamma(\mathbf{v}))$ *and* $\eta(\Gamma(\mathbf{v})) = (\mathbb{J}(\mathbf{v}), \omega_{\Gamma(\mathbf{v})})$.
3. *For $H(T) \in Q_{2n}(A[T])$, we have $\Gamma(H(T))_{T=t} = \Gamma(H(t))$.*
4. *Therefore, $\forall\, \mathbf{v}, \mathbf{w} \in Q_{2n}(S) \quad \zeta_0(\mathbf{v}) = \zeta_0(\mathbf{w}) \Longleftrightarrow \zeta_0(\Gamma(\mathbf{v})) = \zeta_0(\Gamma(\mathbf{w}))$.*

We also record the following obvious observation.

Lemma 5.3. *Suppose A is a commutative noetherian ring with $\dim A = d$. Suppose $n \geq 2$ is an integer. Let $\mathbf{0} \in Q_{2n}(A)$ be the base point and denote $\mathbf{1} := (\Gamma(\mathbf{0})) = (1; 0, \ldots 0; 0, \ldots, 0)$. Then, $\zeta_0(\mathbf{0}) = \zeta_0(\mathbf{1})$.*

Proof. For simplicity, we give the proof for $n = 2$, which we exhibit in the following steps:

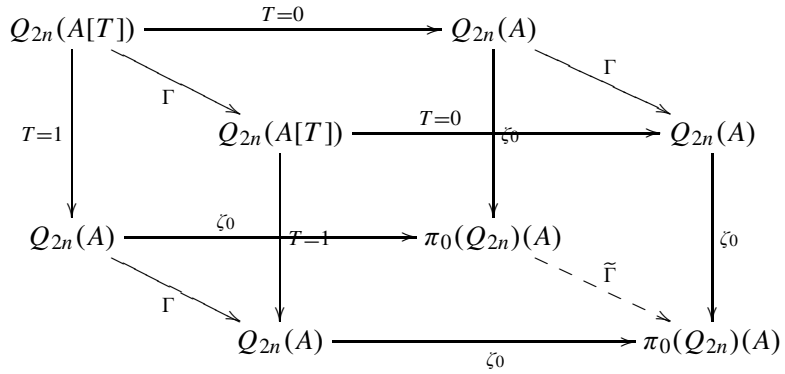
1. By Homotopy $H_1(T) = (0; T, 0; 0, 0) \in Q_4(A[T])$, we have $\zeta_0(0; 0, 0; 0, 0) = \zeta_0(0; 1, 0; 0, 0)$.
2. By Homotopy $H_2(T) = (T; 1, 0; T(1 - T), 0) \in Q_4(A[T])$, we have $\zeta_0(0; 1, 0; 0, 0) = \zeta_0(1; 1, 0; 0, 0)$.
3. By Homotopy $H_3(T) = (1; 1 - T, 0; 0, 0) \in Q_4(A[T])$, we have $\zeta_0(1; 1, 0; 0, 0) = \zeta_0(1; 0, 0; 0, 0)$. The proof is complete.

□

In deed, Γ factors through an involution on $\pi_0(Q_{2n})(A)$, as follows.

Corollary 5.4. *Suppose A is a commutative ring and $n \geq 2$ is an integer. Then, the involution $\Gamma : Q_{2n}(A) \xrightarrow{\sim} Q_{2n}(A)$ induces a bijective map $\widetilde{\Gamma} : \pi_0(Q_{2n})(A) \xrightarrow{\sim} \pi_0(Q_{2n})(A)$, which is also an involution* (meaning, $\widetilde{\Gamma}^2 = 1_{\pi_0(Q_{2n})(A)}$). (The notation $\widetilde{\Gamma}$ will also be among our standard notations throughout this article.)

Proof. It follows from Lemma 5.2, that if $H(T) \in Q_{2n}(A(T))$ is a homotopy from $\mathbf{v}$ to $\mathbf{w}$, then $\Gamma(H(T))$ is a homotopy $\Gamma(\mathbf{v})$ to $\Gamma(\mathbf{w})$. The following commutative diagram of pushout squares,



in $\underline{Sets}$, establishes that $\widetilde{\Gamma}$ is well defined. Clearly, $\widetilde{\Gamma}^2 = 1$. The proof is complete. □

Corollary 5.5. *Suppose A is a commutative ring and $n \geq 2$ is an integer. Suppose $(I, \omega) \in \mathcal{L}O(A, n)$. Assume $f_1, \ldots, f_n \in I$ induce $\omega : A^n \twoheadrightarrow \frac{I}{I^2}$ and $(f_1, \ldots, f_n) = I \cap J$, with $I + J = A$. Then,*

$$\widetilde{\Gamma}(\zeta(I, \omega)) = \zeta(J, \omega_J) \in \pi_0(Q_{2n})(A).$$

Proof. Pick an element $s \in I$ such that $1 - s \in J$. Then, $s(1 - s) \in IJ = (f_1, \ldots, f_n)$. Also, $I = (f_1, \ldots, f_n, s)$ and $J = (f_1, \ldots, f_n, 1 - s)$. Write $s(1 - s) = f_1 g_1 + \cdots + f_n g_n$. With $\mathbf{v} = (s, f_1, \ldots, f_n, g_1, \ldots, g_n) \in Q_{2n}(A)$, we have

$$\widetilde{\Gamma}(\zeta(I, \omega)) = \widetilde{\Gamma}(\zeta_0(\mathbf{v})) = \zeta_0(\Gamma(\mathbf{v})) = \zeta(J, \omega_J).$$

The proof is complete. □

Remark 5.6. We clarify a point that might get lost in the formalisms of (5.4) and (5.5). Suppose $(I, \omega_I) \in \mathcal{L}O(A, n)$ and $\mathbf{u}, \mathbf{v} \in Q_{2n}(A)$, with $\eta(\mathbf{u}) = \eta(\mathbf{v}) = (I, \omega_I)$. We have $\zeta_0(\mathbf{u}) = \zeta_0(\mathbf{v}) = \zeta(I, \omega_I)$ (see [F1]). So, there is a sequence of homotopies $H_1(T), \ldots, H_m(T) \in Q_{2n}(A[T])$, such that, with $H_i(1) = \mathbf{u}_i$, we have $H_1(0) = \mathbf{u} =: \mathbf{u}_0$, $H_i(1) = \mathbf{u}_i = H_{i+1}(0)$ and $H_m(1) = \mathbf{u}_m = \mathbf{v}$. Now, $\Gamma(H_i(T))$ give a sequence of homotopies, starting from $\Gamma(\mathbf{u})$ to $\Gamma(\mathbf{v})$. This means, in the statement of (5.5), $\zeta(J, \omega_J)$ depends on (I, ω_I) only, and is independent of choice of (J, ω_J).

The following is another version of the Moving Lemma 2.4.

Lemma 5.7 (Moving Representation). *Suppose A is a commutative noetherian ring with $\dim A = d$. Suppose $n \geq 2$ is an integer such that $2n \geq d + 1$. Let $K \subseteq A$ be an ideal with height$(K) \geq n$ and $x \in \pi_0(Q_{2n})(A)$. Then, there is a local n-orientation $(J, \omega_J) \in \mathcal{L}O(A, n)$ such that $x = \zeta(J, \omega_J)$, height$(J) \geq n$ and $J + K = A$.*

Proof. It is proved by two successive applications of Moving Lemma 2.4. First, let $x = \zeta(I, \omega_I)$. By (2.4), there is $\mathbf{u} = (s; f_1, \ldots, f_n; g_1, \ldots, g_n)$ such that $\eta(\mathbf{u}) = (I, \omega_I)$. Denote $(I_0, \omega_{I_0}) := \eta(\Gamma(\mathbf{u})) = (\mathbb{J}(\mathbf{u}, \omega_{\Gamma(\mathbf{u})})$. Then, $\tilde{\Gamma}(x) = \zeta(I_0, \omega_{I_0})$.

Now, we apply Moving Lemma 2.4, to (I_0, ω_{I_0}) and K. There is $\mathbf{v} = (S; F_1, \ldots, F_n; G_1, \ldots, G_n) \in Q_{2n}(A)$, such that $\eta(\mathbf{v}) = (I_0, \omega_{I_0})$, and with $J = \mathbb{J}(\mathbf{v})$, we have *height*$(J) \geq n$ and $J + K = A$. Now, $x = \tilde{\Gamma}(\tilde{\Gamma}(x)) = \tilde{\Gamma}(\zeta(I_0, \omega_{I_0})) = \zeta(J, \omega_J)$, where $\omega_J := \omega_{\Gamma(\mathbf{v})}$. The proof is complete. □

We record the following useful lemma.

Lemma 5.8. *Suppose A is a commutative noetherian ring with $\dim A = d$. Suppose $n \geq 2$ is an integer. Suppose $\mathbf{v} = (s; f_1, \ldots, f_n; g_1, g_1, \ldots, g_n) \in Q_{2n}(A)$. Then, $\zeta_0(\mathbf{v}) = \zeta_0(\mathbf{0}) \iff \zeta_0(\Gamma(\mathbf{v})) = \zeta_0(\mathbf{0})$.*

Proof. By Lemma 5.2 (4), and Lemma 5.3.

$$\zeta_0(\mathbf{v}) = \zeta_0(\mathbf{0}) \iff \zeta_0(\Gamma\mathbf{v}) = \zeta_0(\mathbf{1}) = \zeta_0(\mathbf{0}).$$

□

As a consequence of the above, we formulate the following homotopy version of Subtraction Principle.

Theorem 5.9. *Suppose A is a commutative noetherian ring with $\dim A = d$ and $n \geq 2$ is an integer. Let I, J be two ideals in A such that $I + J = A$ and $IJ = (f_1, \ldots, f_n)$. Let $(I, \omega_I), (J, \omega_J) \in \mathcal{L}O(A, n)$ be induced by $f_1, \ldots, f_n$. Then, $\zeta(I, \omega_I)) = \zeta_0(\mathbf{0}) \iff \zeta(J, \omega_J) = \zeta_0(\mathbf{0})$.*

Proof. By local checking, it follows that

$$I = (f_1, \ldots, f_n) + I^2, \quad \text{and} \quad J = (f_1, \ldots, f_n) + J^2.$$

Since $I + J = A$, there is an element $s \in I$ such that $1 - s \in J$. So, $s(1 - s) \in IJ$ and hence

$$\sum_{i=1}^{n} f_i g_i + s(s-1) = 0 \quad \text{for some } g_1, \ldots, g_n \in A.$$

Therefore,

$$\mathbf{v} := (s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in Q_{2n}(A).$$

We have $\zeta_0(\mathbf{v}) = \zeta(I, \omega_I)$ and $\zeta_0(\Gamma(\mathbf{v})) = \zeta(J, \omega_J)$. Now, the proof follows from (5.8). □

6. The group structure on $\pi_0(Q_{2n})(A)$

In this section, we establish a group structure on the set $\pi_0(Q_{2n})(A)$, when $2n \geq \dim A + 2$ and A is a regular ring over a field k, with $1/2 \in k$.

We start with the following basic ingredient of the group structure.

Definition 6.1. *Let A be a commutative noetherian ring and $n \geq 2$ be an integer.* (Refer to notations η, ζ_0, ζ in diagram 4.) *Let $(I, \omega_I), (J, \omega_J) \in \mathcal{L}O(A, n)$ be such that $I + J = A$. Let $\omega := \omega_I \star \omega_J : A^n \twoheadrightarrow \frac{IJ}{(IJ)^2}$ be the unique surjective map induced by ω_I, ω_J. We define a pseudo-sum*

$$(I, \omega_I)\hat{+}(J, \omega_J) := \zeta(IJ, \omega) \in \pi_0(Q_{2n})(A).$$

Also, for $\mathbf{u}, \mathbf{v} \in Q_{2n}(A)$ with $\mathbb{I}(\mathbf{u}) + \mathbb{I}(\mathbf{v}) = A$, define pseudo-sum

$$\mathbf{u}\hat{+}\mathbf{v} := \eta(\mathbf{u})\hat{+}\eta(\mathbf{v}) \in \pi_0(Q_{2n})(A).$$

The rest of this section is devoted to establish that this pseudo sum respects homotopy and extends to $\pi_0(Q_{2n})(A)$, when A is a regular ring over a field k, with $1/2 \in k$. The following is an obvious corollary.

Corollary 6.2. *Let A be a commutative noetherian ring with $\dim A = d$ and $n \geq 2$ be an integer, with $2n \geq d + 3$. Suppose $(I, \omega_I), (J, \omega_J) \in \mathcal{LO}(A, n)$ and $I + J = A$. Let $\mathbf{u} = (s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in Q_{2n}(A)$ and $\mathbf{v} = (S; F_1, \ldots, F_n; G_1, \ldots, G_n) \in Q_{2n}(A)$ be such that $\eta(\mathbf{u}) = (I, \omega_I)$ and $\eta(\mathbf{v}) = (J, \omega_J)$. Write $\eta(\Gamma(\mathbf{u})) = (I_1, \omega_{I_1}))$ and $\eta(\Gamma(\mathbf{v})) = (J_1, \omega_{J_1}))$. We have $I + I_1 = J + J_1 = A$, and further assume $J + I_1 = I + J_1 = A$. (Such choices of $\mathbf{u}, \mathbf{v}$ would be available, because $2n \geq d + 1$.) Then,*

$$\tilde{\Gamma}((I, \omega_I)\hat{+}(J, \omega_J)) = (I_1, \omega_{I_1})\hat{+}(J_1, \omega_{J_1}).$$

Proof. From addition principle ([BK, Theorem 5.6])

$$I \cap I_1 \cap J \cap J_1 = (U_1, \ldots, U_n) \quad \ni \quad U_i - f_i \in (II_1)^2, \;\; U_i - F_i \in (JJ_1)^2.$$

Let $\omega : A^n \twoheadrightarrow \frac{IJ}{(IJ)^2}$, $\omega' : A^n \twoheadrightarrow \frac{I_1J_1}{(I_1J_1)^2}$ be the surjective maps induced by $U_1, \ldots, U_n$. If follows $\omega = \omega_I \star \omega_J$ and $\omega' = \omega_{I_1} \star \omega_{J_1}$. So, by corollary 5.5,

$$\tilde{\Gamma}((I, \omega_I)\hat{+}(J, \omega_J)) = \tilde{\Gamma}(\zeta(IJ, \omega)) = \zeta(I_1J_1, \omega') = (I_1, \omega_{I_1})\hat{+}(J_1, \omega_{J_1}).$$

The proof is complete. □

Remark. Note that the involution operations $\Gamma : Q_{2n}(A) \longrightarrow Q_{2n}(A)$ does not factor through a map $\mathcal{LO}(A, n) \longrightarrow \mathcal{LO}(A, n)$.

Now we define a pseudo-difference in the spirit of (6.1).

Definition 6.3. *Suppose A is a commutative ring and $n \geq 2$ be an integer. Suppose $(K, \omega_K), (I, \omega_I) \in \mathcal{L}O(A, n)$. Assume that there is $\mathbf{u} = (s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in Q_{2n}(A)$ such that $\eta(\mathbf{u}) = (I, \omega_I)$ and with $J = \mathbb{J}(\mathbf{u})$, $J + K = A$. Therefore,*

$$(f_1, \ldots, f_n) = I \cap J \quad \ni \quad I + J = K + J = A.$$

Let $\omega_J : A^n \twoheadrightarrow \frac{J}{J^2}$ be the surjective map induced by $f_1, \ldots, f_n$. Then, define the pseudo-difference

$$(K, \omega_K)\hat{-}(I, \omega_I) := (K, \omega_K)\hat{+}(J, \omega_J) \in \pi_0(Q_{2n})(A).$$

We remark: (1) A priori, the pseudo-difference depends on the choice of J. There is no conditions on *height*(I), *height*(J), *height*(K); nor did we assume $I + K = A$. (2) By Moving Lemma 2.4, such choices $\mathbf{u} \in Q_{2n}(A)$ would be available if $2n \geq \dim A + 1$ and *height*$(K) \geq n$.

Subsequently, under additional hypotheses, we would first prove that the definition (6.3) of pseudo difference does not depend of the choice of (J, ω_J). Then, we prove that the pseudo difference is homotopy invariant with respect to either coordinate. A key to such proofs would be the following lemma that combines (*i.e. "adds"*) homotopies. (*For basic element theory and the definition of generalized dimension functions, we refer to* [M1].)

Lemma 6.4. *Suppose A is a commutative noetherian ring with* $\dim A = d$ *and $n \geq 2$ is an integer with $2n \geq d+2$. Consider a homotopy*

$$H(T) = (Z, \varphi_1, \ldots, \varphi_n, \gamma_1, \ldots, \gamma_n) \in Q_{2n}(A[T]).$$

Write $\eta(H(0)) = (K_0, \omega_{K_0})$ and $\eta(H(1)) = (K_1, \omega_{K_1})$. Further suppose $(J, \omega_J) \in \mathcal{L}O(A, n)$ such that $K_0 + J = K_1 + J = A$ and height$(J) \geq n$. Then, there is a homotopy $\mathcal{H}(T) \in Q_{2n}(A[T])$ such that $\eta(\mathcal{H}(0)) = (K_0J, \omega_{K_0J})$ and $\eta(\mathcal{H}(1)) = (K_1J, \omega_{K_1J})$, where, for $i = 0, 1$ $\omega_{K_iJ} := \omega_{K_i} \star \omega_J : A^n \twoheadrightarrow \frac{K_iJ}{(K_iJ)^2}$.

Proof. Write $Y = 1 - Z$. Then, $\mathbb{J}(H(T)) = (\varphi_1, \ldots, \varphi_n, Y)$. Write

$$\mathscr{P} = \{\wp \in \text{Spec}\,(A[T]) : YT(1 - T) \notin \wp, J \subseteq \wp\}.$$

There is a generalized dimension function $\delta : \mathscr{P} \longrightarrow \mathbb{N}$ such that $\forall\ \wp \in \mathscr{P}$, $\delta(\wp) \leq \dim\left(\frac{A[T]}{JA[T]}\right) \leq d + 1 -$ *height*$(J) \leq d + 1 - n \leq n - 1$. Further, $(\varphi_1, \ldots, \varphi_n, Y^2T(1 - T))$ is a basic element in $A[T]^{n+1}$, on $\mathscr{P}$. Therefore, there are polynomials $\lambda_1, \ldots, \lambda_n \in A[T]$ such that, with $\varphi_i' = \varphi_i + \lambda_i Y^2 T(1 - T)$ for $i = 1, \ldots, n$, we have

$$(\varphi_1', \varphi_2', \ldots, \varphi_n') \quad \text{is basic on } \mathscr{P}.$$

We have $\boldsymbol{\forall\ i = 1, \ldots, n\ \varphi_i(0) = \varphi_i'(0),\ \varphi_i(1) = \varphi_i'(1)}$. We compute

$$Z(1 - Z) = Y(1 - Y) = \sum_{i=1}^{n} \varphi_i \gamma_i = \sum_{i=1}^{n} \varphi_i' \gamma_i - Y^2 T(1 - T) \sum_{i=1}^{n} \lambda_i \gamma_i$$

So,

$$Y = \sum_{i=1}^{n} \varphi_i' \gamma_i - Y^2 T(1-T)\mu + Y^2 \quad \text{where} \quad \mu = \sum_{i=1}^{n} \lambda_i \gamma_i.$$

Write $M = \frac{\mathbb{J}(H(T))}{(\varphi_1', \varphi_2', \ldots, \varphi_n')}$. Use "overline" to indicate images in M. We intend to repeat the proof of Nakayama's Lemma and we have

$$\begin{pmatrix} \overline{\varphi_1} \\ \overline{\varphi_2} \\ \cdots \\ \overline{\varphi_n} \\ \overline{Y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\lambda_1 YT(1-T) \\ 0 & 0 & \cdots & 0 & -\lambda_2 YT(1-T) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -\lambda_n YT(1-T) \\ 0 & 0 & 0 & 0 & Y - YT(1-T)\mu \end{pmatrix} \begin{pmatrix} \overline{\varphi_1} \\ \overline{\varphi_2} \\ \cdots \\ \overline{\varphi_n} \\ \overline{Y} \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 & \lambda_1 YT(1-T) \\ 0 & 1 & \cdots & 0 & \lambda_2 YT(1-T) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \lambda_n YT(1-T) \\ 0 & 0 & 0 & 0 & 1 - Y + YT(1-T)\mu \end{pmatrix} \begin{pmatrix} \overline{\varphi_1} \\ \overline{\varphi_2} \\ \cdots \\ \overline{\varphi_n} \\ \overline{Y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix}$$

Multiplying by the adjoint matrix and computing the determinant, with $Y' = Y - YT(1-T)\mu$, we have

$$(1 - Y')\mathbb{J}(H(T)) \subseteq (\varphi_1', \varphi_2', \ldots, \varphi_n').$$

We have $\boldsymbol{Y'(0) = Y(0) = 1 - Z(0)}$, $\boldsymbol{Y'(1) = Y(1) = 1 - Z(1)}$. Further,

$$Y'(1 - Y') = \sum_{i=1}^{n} \varphi' \gamma_i' \quad \text{for some polynomials} \quad \gamma_1', \ldots, \gamma_n' \in A[T]$$

$$H'(T) = (Y', \varphi_1', \ldots, \varphi_n'; \gamma_1', \ldots, \gamma_n') \in Q_{2n}(A[T]).$$

We have

$$\mathbb{J}(H(Y)) = (\varphi_1, \ldots, \varphi_n, Y) = (\varphi_1', \ldots, \varphi_n', Y) = (\varphi_1', \ldots, \varphi_n', Y') = \mathbb{I}(H'(T)).$$

Claim

$$\mathbb{J}(H'(T)) + JA[T] = A[T]. \quad \text{i.e.} \quad (\varphi_1', \ldots, \varphi_n', 1 - Y') + JA[T] = A[T].$$

To see this, let

$$\mathbb{J}(H'(T)) + JA[T] \subseteq \wp \in \text{Spec}\,(A[T])$$

1. If $Y \in \wp$ then $(\varphi_1', \ldots, \varphi_n', Y) = (\varphi_1', \ldots, \varphi_n', Y') \subseteq \wp$. So, $Y' \in \wp$, which is impossible, since $1 - Y' \in \wp$. ***So, $\wp \in D(Y)$.***
2. Since $\wp \in D(Y)$ and since $(\varphi_1', \ldots, \varphi_n')$ is unimodular of $\mathscr{P}$, we must have $T(1-T) \in \wp$.
3. Now, $T \in \wp$ implies,

$$\mathbb{J}(H'(0)) + J = (\varphi_1'(0), \ldots, \varphi_n'(0), 1 - Y'(0)) + J = (\varphi_1(0), \ldots, \varphi_n(0), 1 - Y(0)) + J$$
$$= (\varphi_1(0), \ldots, \varphi_n(0), Z(0)) + J = K_0 + J = A \subseteq \wp,$$

which is impossible.

4. Likewise, $1 - T \in \wp$ implies,

$$\begin{aligned}\mathbb{J}(H'(1)) + J &= (\varphi_1'(1), \dots, \varphi_n'(1), 1 - Y'(1)) + J = (\varphi_1(1), \dots, \varphi_n(1), 1 - Y(1)) + J \\ &= (\varphi_1(1), \dots, \varphi_n(1), Z(1)) + J = K_1 + J = A \subseteq \wp\end{aligned}$$

This is also impossible.

This establishes the claim. So,

$$(\varphi_1', \dots, \varphi_n') = \mathbb{I}(H'(T)) \cap \mathbb{J}(H'(T)) \quad \text{where} \quad \mathbb{J}(H'(T)) = (\varphi_1', \dots, \varphi_n', 1 - Y').$$

Let

$$\Omega_{\mathbb{J}(H'(T))} : A[T]^n \twoheadrightarrow \frac{\mathbb{J}(H'(T))}{\mathbb{J}(H'(T))^2} \quad \text{be induced by} \quad \varphi_1', \dots, \varphi_n'.$$

Extend $\omega_J : A^n \twoheadrightarrow \frac{J}{J^2}$ to a surjective map $\omega_{JA[T]} : A[T]^n \twoheadrightarrow \frac{JA[T]}{J^2A[T]}$. Let

$$\Omega := \Omega_{\mathbb{J}(H'(T))} \star \omega_{JA[T]} : A[T]^n \twoheadrightarrow \frac{J\mathbb{J}(H'(T))}{J^2\mathbb{J}(H'(T))^2} \quad \text{be induced by} \quad \Omega_{\mathbb{J}(H'(T))}, \omega_{JA[T]}.$$

Now, Ω has a lift $\mathcal{H}(T) \in Q_{2n}(A[T])$. Specializing at $T = 0$ and $T = 1$, we have

$$\eta(\mathcal{H}(0)) = (K_0J, \omega_{K_0J}), \quad \eta(\mathcal{H}(1)) = (K_1J, \omega_{K_1J}).$$

The proof is complete. □

Now we proceed to prove, in several propositions, that the pseudo-difference (6.3) is well defined and homotopy invariant.

Proposition 6.5. *Suppose A is a regular ring over a field k, with $1/2 \in k$, with $\dim A = d$. Let $n \geq 2$ be an integer, such that $2n \geq d + 2$. As in (6.3), let $(K, \omega_K), (I, \omega_I) \in \mathcal{L}O(A, n)$ be given.*

*Let $\mathbf{u} = (s, f_1, \dots, f_n; g_1, \dots, g_n) \in Q_{2n}(A) \in Q_{2n}(A)$ be such that, with $\eta(\mathbf{u}) = (I, \omega_I)$ and $\eta(\Gamma(\mathbf{u})) = (J, \omega_J)$, $K + J = A$. Likewise, let $\mathbf{v} = (S, F_1, \dots, F_n; G_1, \dots, G_n) \in Q_{2n}(A)$ be such that $\eta(\mathbf{v}) = (I, \omega_I)$, and with $\eta(\Gamma(\mathbf{v})) = (L, \omega_L)$, $K + L = A$. Assume **height$(K) \geq n$**. Then,*

$$(K, \omega_K)\hat{+}(J, \omega_J) = (K, \omega_K)\hat{+}(L, \omega_L) \in \pi_0(Q_{2n}(A).$$

(Recall form (6.3), there is no restriction on *height*(I), *height*(J) and, nor did we assume $I + K = A$.)

Proof. By hypotheses, $K + J = K + L = A$. By Corollary 5.5,

$$\widetilde{\Gamma}(\zeta(I, \omega_I)) = [\zeta(J, \omega_J)] = [\zeta(L, \omega_L)]. \text{ Hence } \zeta_0(\Gamma(\mathbf{u})) = \zeta_0(\Gamma(\mathbf{v})).$$

By Corollary 4.3, there is a homotopy

$$H(T) = (Z(T); \varphi_1(T), \dots, \varphi_n(T); \gamma_1(T), \dots, \gamma_n(T)) \in Q_{2n}(A[T])$$

such that $H(0) = \Gamma(\mathbf{u})$, $H(1) = \Gamma(\mathbf{v})$. By the Homotopy Lemma 6.4 there is a homotopy $\mathcal{H}(T) \in Q_{2n}(A[T])$ such that $\eta(\mathcal{H}(0)) = (KJ, \omega_K \star \omega_J)$ and $\eta(\mathcal{H}(1)) = (KL, \omega_K \star \omega_L)$. This establishes,

$$(K, \omega_K)\hat{+}(J, \omega_J) = (K, \omega_K)\hat{+}(L, \omega_L) \in \pi_0(Q_{2n}(A).$$

This completes the proof. □

Corollary 6.6. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let $n \geq 2$ be an integer, such that $2n \geq d + 2$. Let $(K, \omega_K), (I, \omega_I) \in \mathcal{LO}(A, n)$. Assume **height$(K) \geq n$**. Then, the pseudo-difference* (6.3)

$$(K, \omega_K)\hat{-}(I, \omega_I) \in \pi_0(Q_{2n}(A)) \quad \textit{is well defined.}$$

Proof. This is immediate from Proposition 6.5. □

Now we prove that the pseudo-difference is homotopy invariant, with respect to the (I, ω_I)-coordiante.

Proposition 6.7. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let $n \geq 2$ is an integer, with $2n \geq d+2$. Let $(K, \omega_K), (I_0, \omega_{I_0}), (I_1, \omega_{I_1}) \in \mathcal{L}O(A, n)$ and **height$(K) \geq n$**. Suppose $\zeta(I_0, \omega_{I_0}) = \zeta(I_1, \omega_{I_1}) \in \pi_0(Q_{2n}(A))$. Then,*

$$(K, \omega_K)\hat{-}(I_0, \omega_{I_0}) = (K, \omega_K)\hat{-}(I_1, \omega_{I_1})\pi_0(Q_{2n}(A)).$$

In other words, pseudo differences (defined in 6.3) *are homotopy invariant, with respect to the latter coordinate.*

Proof. By Moving Lemma 2.4, we can find $\mathbf{u}_0, \mathbf{u}_1 \in Q_{2n}(A)$ such that $\eta(\mathbf{u}_0) = (I_0, \omega_{I_0})$, $\eta(\mathbf{u}_1) = (I_1, \omega_{I_1})$, and with $J_0 = \mathbb{J}(\mathbf{u}_0)$, $J_1 = \mathbb{J}(\mathbf{u}_1)$, $K + J_0 = K + J_1 = A$. For $i = 0, 1$, let $\omega_{J_i} = \omega_{\Gamma(\mathbf{u}_i)} : A^n \twoheadrightarrow \frac{J_i}{J_i^2}$. By hypothesis, $\zeta(I_0, \omega_{I_0}) = \zeta(I_1, \omega_{I_1})$ and hence

$$\zeta_0(\Gamma(\mathbf{u}_0)) = \zeta(J_0, \omega_{J_0}) = \widetilde{\Gamma}(\zeta(I_0, \omega_{I_0})) = \widetilde{\Gamma}(\zeta(I_1, \omega_{I_1})) = \zeta(J_1, \omega_{J_1}) = \zeta_0(\Gamma(\mathbf{u}_1)).$$

By Corollary 4.3, there is a homotopy $H(T) \in Q_{2n}(A[T])$ such that $H(0) = \Gamma(\mathbf{u}_0)$ and $H(1) = \Gamma(\mathbf{u}_1)$. Since $K + J_0 = K + J_1 = A$, by Lemma 6.4, there is a homotopy $\mathcal{H}(T) \in Q_{2n}(A[T])$ such that $\eta(\mathcal{H}(0)) = (KJ_0, \omega_K \star \omega_{J_0})$ and $\eta(\mathcal{H}(1)) = (KJ_0, \omega_K \star \omega_{J_1})$. Therefore,

$$(K, \omega_K)\hat{+}(J_0, \omega_{J_0}) = (K, \omega_K)\hat{+}(J_1, \omega_{J_1})$$

and hence

$$(K, \omega_K)\hat{-}(I_0, \omega_{I_0}) = (K, \omega_K)\hat{-}(I_1, \omega_{I_1}).$$

The proof is complete. □

We extend the definition of pseudo difference to $\pi_0(Q_{2n})(A)$ as follows.

Corollary 6.8. *Suppose A is a regular ring over a field k, with $1/2 \in k$, with $\dim A = d$. Let $n \geq 2$ is an integer, with $2n \geq d+2$. Let $(K, \omega_K) \in \mathcal{L}O(A, n)$ and **height$(K) \geq n$**. Then, there is a well defined set theoretic map*

$$\pi_0(Q_{2n})(A) \longrightarrow \pi_0(Q_{2n})(A) \quad \text{sending} \quad \zeta(I, \omega_I) \mapsto (K, \omega_K)\hat{-}(I, \omega_I).$$

Proof. Immediate from Proposition 6.7. □

Now, we extend the pseudo-difference to $\pi_0(Q_{2n})(A) \times \pi_0(Q_{2n})(A)$.

Theorem 6.9. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let $n \geq 2$ is an integer, with $2n \geq d+2$. Then, there is a set theoretic map*

$$\Theta : \pi_0(Q_{2n})(A) \times \pi_0(Q_{2n})(A) \longrightarrow \pi_0(Q_{2n})(A)$$

*such that, for $(K, \omega_K) \in \mathcal{L}O(A, n)$, with **height$(K) \geq n$**, and $(I, \omega_I) \in \mathcal{L}O(A, n)$,*

$$\Theta(\zeta(K, \omega_K), \zeta(I, \omega_I)) = (K, \omega_K)\hat{-}(I, \omega_I). \tag{9}$$

Proof. Suppose $x \in \pi_0(Q_{2n})(A)$. By the Moving Lemma 5.7, we can write $x = \zeta(K, \omega_K)$, with $height(K) \geq n$. Therefore, if well defined, the Equation 9 applies to all $(x, y) \in \pi_0(Q_{2n})(A) \times \pi_0(Q_{2n})(A)$.

Let $x, y \in \pi_0(Q_{2n})(A)$. We can write $x = \zeta(K, \omega_K)$, with $height(K) \geq n$ and $y = \zeta(I, \omega_I)$. Define,

$$\Theta(x, y) = (K, \omega_K)\hat{-}(I, \omega_I).$$

We need to prove that, if $x = \zeta(K, \omega_K) = \zeta(K', \omega_{K'})$, with $height(K) \geq n$ and $height(K') \geq n$, then

$$(K, \omega_K)\hat{-}(I, \omega_I) = (K', \omega_{K'})\hat{-}(I, \omega_I).$$

Again, by Moving Lemma 2.4, there is $\mathbf{u} = (s; f_1, f_2, \ldots, f_n; g_1, \ldots, g_n) \in Q_{2n}(A)$ such that $\eta(\mathbf{u}) = (I, \omega_I)$, and with $\eta(\Gamma(\mathbf{u})) = (J, \omega_J)$, we have $J + KK' = A$. Since $x = \zeta(K, \omega_K) = \zeta(K', \omega_{K'})$, by Corollary 4.3, there is a homotopy $H(T) \in Q_{2n}(A[T])$ such that $\eta(H(0)) = (K, \omega_K)$ and $\eta(H(1)) = (K', \omega_{K'})$. By Lemma 6.4, there is a homotopy $\mathcal{H}(T) \in Q_{2n}(A[T])$ such that $\eta(\mathcal{H}(0)) = (KJ, \omega_K \star \omega_J)$ and $\eta(\mathcal{H}(1)) = (K'J, \omega_{K'} \star \omega_J)$. Therefore,

$$(K, \omega_K)\hat{+}(J, \omega_J) = \zeta(KJ, \omega_K \star \omega_J) = \zeta(K'J, \omega_{K'} \star \omega_J) = (K', \omega_{K'})\hat{+}(J, \omega_J).$$

Therefore, by definition,

$$(K, \omega_K)\hat{-}(I, \omega_I) = (K', \omega_{K'})\hat{-}(I, \omega_I).$$

This completes the proof. □

Finally, we are ready to define the group structure on $\pi_0(Q_{2n})(A)$.

Definition 6.10. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let $n \geq 2$ is an integer, with $2n \geq d + 2$. Then, for $x, y \in \pi_0(Q_{2n})(A)$, the association*

$$(x, y) \mapsto \Theta(x, \widetilde{\Gamma}(y))$$

is a well defined binary operation on $\pi_0(Q_{2n})(A)$, where Θ is as in (6.9). This operation is well defined because so are Θ and $\widetilde{\Gamma}$ (see Corollary 5.4). We denote

$$x + y := \Theta(x, \widetilde{\Gamma}(y))$$

This binary operation will be referred to an addition.

With the help of the Moving Lemma 2.4, the addition operation on $\pi_0(Q_{2n})(A)$ can be described in a more direct manner, as follows.

Lemma 6.11. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let $n \geq 2$ is an integer, with $2n \geq d + 2$. Let $x, y \in \pi_0(Q_{2n})(A)$. By the Moving Lemma 2.4, $x = \zeta(K, \omega_K)$ and $y = \zeta(I, \omega_I)$, for some $(K, \omega_K), (I, \omega_I) \in \mathcal{L}O(A, n)$, such that $I + K = A$ and $height(K) \geq n$. Then,*

$$x + y = (K, \omega_K)\hat{+}(I, \omega_I) \quad \text{as in Definition (6.1).}$$

Proof. Let $\mathbf{u} \in Q_{2n}(A)$ be such that $\eta(\mathbf{u}) = (I, \omega_I)$ and write $\eta(\mathbf{u}) = (J, \omega_J)$. Then, $\widetilde{\Gamma}(\zeta(I, \omega_I)) = \zeta(J, \omega_J)$. By Definition 6.10,

$$x + y = \Theta(x, \widetilde{\Gamma}(y)) = \Theta(x, \zeta(J, \omega_J)) = (K, \omega_K)\hat{-}(J, \omega_J) = (K, \omega_K)\hat{+}(I, \omega_I).$$

This completes the proof. □

The following is a final statement on the group structure on $\pi_0(Q_{2n})(A)$.

Theorem 6.12. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let $n \geq 2$ is an integer, with $2n \geq d + 2$. Then, the addition operation on $\pi_0(Q_{2n})(A)$, defined in (6.10), endows a structure of an abelian group on $\pi_0(Q_{2n})(A)$.*

Proof. First, $\mathbf{0} = \zeta(A, \omega_A) = \zeta_0(0; 0, \ldots, 0; 0, \ldots, 0) = \zeta_0(1; 0, \ldots, 0; 0, \ldots, 0)$ acts as the additive identity of this addition. Given $x, y, z \in \pi_0(Q_{2n})(A)$, by applications of the Moving Lemma 2.4, we can write

$$x = \zeta(K, \omega_K),\ y = \zeta(I, \omega_I)\ z = \zeta(J, \omega_J) \quad \ni\ K + I = K + J = I + J = A$$

and $height(K) \geq n$, $height(I) \geq n$, $height(J) \geq n$. By Lemma 6.11, we have the following.

$$(x+y)+z = ((K,\omega_K)\hat{+}(I,\omega_I))\hat{+}(J,\omega_J) = x+(y+z).$$

So, the associativity holds. Further,

$$x+y = (K,\omega_K)\hat{+}(I,\omega_I) = (I,\omega_I)\hat{+}(K,\omega_K) = y+x.$$

So, the commutativity holds. Also,

$$x+\widetilde{\Gamma}(x) = \Theta(x,x) = 0.$$

So, x has an additive inverse. This completes the proof. □

6.1 The Euler class groups

In this subsection we compare the Euler class groups $E^n(A)$ and $\pi_0(Q_{2n})(A)$. Refer to Section A for the definition of Euler class groups and some notations used in this section. First, we define a map $\rho : E^n(A) \longrightarrow \pi_0(Q_{2n})(A)$, as follows.

Definition 6.13. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let $n \geq 2$ be an integer, with $2n \geq d+2$. Then, the restriction of the map $\zeta : \mathcal{L}O(A,n) \longrightarrow \pi_0(Q_{2n})(A)$ to $\mathcal{L}O^c(A,n)$ (see* (Definition A.1) *for this notation) defines a set theocratic map $\mathcal{L}O^c(A,n) \longrightarrow \pi_0(Q_{2n})(A)$. Since $\pi_0(Q_{2n})(A)$ has the structure of an abelian group (6.11), this extends to a group homomorphism $\rho_0 : \mathbb{Z}(\mathcal{L}O^c(A,n)) \longrightarrow \pi_0(Q_{2n})(A)$. Then, ρ_0 induces a surjective homomorphism $\rho : E^n(A) \twoheadrightarrow \pi_0(Q_{2n})(A)$.*

Proof. That ρ_0 factors through a homomorphism $\rho : E^n(A) \twoheadrightarrow \pi_0(Q_{2n})(A)$ is obvious. For $x \in \pi_0(Q_{2n})(A)$, by Moving Lemma 2.4, $x = \zeta(I,\omega_I)$ for some $(I,\omega_I) \in \mathcal{L}O(A,n)$, with $height(I) \geq n$. It follows, by (6.12), $x = \rho(\overline{\varepsilon}(I,\omega_I))$. This completes the proof. □

Remark 6.14. It follows from [BS2, Theorem 4.2], if ρ in (6.13) is an isomorphism and $2n \geq d+3$, then for $(I,\omega_I) \in \mathcal{L}O(A,n)$, with $height(I) \geq n$, $\zeta(I,\omega_I) = \mathbf{0}$ implies ω_I lifts to a surjective map $A^n \twoheadrightarrow I$.

Conversely, the homomorphism ρ in (6.13) is clearly an isomorphism, if for $(I,\omega_I) \in \mathcal{L}O(A,n)$, and $height(I) \geq n$, $\zeta(I,\omega_I) = 0$ implies ω_I lifts to a surjective map $A^n \twoheadrightarrow I$. This would be case, in the following case.

Theorem 6.15. *Suppose k is an infinite perfect field, with $1/2 \in k$ and A is an essentially smooth ring over k, with $\dim A = d$. Let $n \geq 2$ be an integer, with $2n \geq d+3$. Then, the homomorphism ρ in (6.13) is an isomorphism.*

Proof. We only need to prove that ρ is injective. Let $\rho(x) = 0$ for some $x \in E^n(A)$. We can write $x = \overline{\varepsilon}(I,\omega_I)$, for some $(I,\omega_I) \in \mathcal{L}O(A,n)$, with $height(I) \geq n$. Let $\mathbf{u} \in Q_{2n}(A)$ be such that $\eta(\mathbf{u}) = (I,\omega_I)$. By (4.3), there is a homotopy $H(T) \in Q_{2n}(A[T])$ such that $H(0) = \mathbf{1}$ and $H(1) = \mathbf{u}$. Write $H(T) = (Z; f_1,\ldots,f_n; g_1,\ldots,g_n)$. By Moving Lemma, we can modify $f_1,\ldots,f_n$ by multiples of $T(1-T)Z^2$ and assume $height(\mathbb{I}(H(T))) \geq n$. Write $\eta(H(T)) = (\mathcal{I},\omega)$. Then, $\eta(H(0)) = \eta(\mathbf{1}) = (A,*)$, where $*$ is the trivial map. Since $\mathcal{I}+TA[T] = A[T]$, we have $\frac{\mathcal{I}}{T\mathcal{I}} \xrightarrow{\sim} \mathcal{I}(0) = A$. Let $\lambda : A^n \twoheadrightarrow A$ be the map given by $(1,0,\ldots,0)$. Then, ω and λ combines to give surjective homomorphism $\varphi_0 : A[T]^n \twoheadrightarrow \frac{\mathcal{I}}{T\mathcal{I}^2}$. By [BK, Theorem 4.13], φ_0 lifts to a surjective map $\varphi : A[T]^n \twoheadrightarrow \mathcal{I}$. Specializing at $T=1$, we have $\varphi(1) : A^n \twoheadrightarrow \mathcal{I}(0) = I$ is a lift of ω_I. Therefore $\overline{\varepsilon}(I,\omega_I) = 0 \in E^n(A)$. This completes the proof. □

A. Preliminaries on Euler class groups

For a regular ring A with $\dim A = d$, Nori outlined a definition of an alternate obstruction group $E^d(A)$, of codimension d-cycles, which was expected to coincide with $\pi_0(Q_{2d})(A)$. Generators of the groups were $S = \{(m,\omega_m) \in \mathcal{L}O(A,d) : m \in \max(A)\}$. By some innovative analogies, for each co-dimension $n \geq 0$, Bhatwadekar and Sridharan [BS2] defined an obstructions group, where A was assumed to be any noetherian commutative ring. These groups have come to be known as Euler Class Groups. In this section we modify the definition in [BS2], to avoid some superfluous elements. (*In the sequel, for a set S, the free abelian group generated by S will be denoted by $\mathbb{Z}(S)$*).

Definition A.1. *Suppose A is a noetherian commutative ring, with $\dim A = d$ and $n \geq 0$ is an integer. Let $\mathcal{L}O^c(A,n) = \{(I,\omega_I) \in \mathcal{L}O(A,n) : V(I)$ is connected and $height(I) \geq n\}$.*

Suppose $(I,\omega_I) \in \mathcal{L}O(A,n)$ and $I = \cap_{i=1}^m I_i$ be a decomposition, where $V(I_i) \subseteq \mathrm{Spec}\,(A)$ are connected. For $i = 1,\ldots,m$, the orientation (I,ω_I) induce orientations $(I_i,\omega_{I_i}) \in \mathcal{L}O(A,n)$. Denote $\varepsilon(I,\omega_I) = \sum_{i=1}^m (I_i,\omega_{I_i}) \in \mathbb{Z}(\mathcal{L}O^c(A,n))$. Recall, an orientation $(I,\omega_I) \in \mathcal{L}O(A,n)$ is called global, if ω_I lifts to a surjective map $A^n \twoheadrightarrow I$. Let $\mathscr{R}^n(A)$ denote the subgroup of $\mathbb{Z}(\mathcal{L}O^c(A,n))$, generated by the set $\{\varepsilon(I,\omega_I) : (I,\omega_I)$ is global and $height(I) \geq n\}$. Define

$$E^n(A) = \frac{\mathbb{Z}(\mathcal{L}O^c(A,n))}{\mathscr{R}^n(A)}$$

Images of $\varepsilon(I,\omega_I)$ in $E^n(A)$ with be denoted by $\overline{\varepsilon}(I,\omega_I)$.

The definition of the Euler class group in [BS2], which we denote by $\mathcal{E}^n(A)$, differs a little from the above, which we recall.

Given an ideal I, two local orientations $(I,\omega),(I,\omega') \in \mathcal{L}O(A,n)$ were defined to be equivalent, if $\omega' = \omega\sigma$ for some elementary matrix $\sigma \in EL_n(A)$. The equivalence class of (I,ω) is denoted by $(I,[\omega])$. Let $\mathcal{L}O_0^c(A,n) = \{(I,[\omega_I]) : (I,\omega_I) \in \mathcal{L}O(A,n) : V(I)$ is connected and $height(I) \geq n\}$. Again, for $(I,\omega_I) \in \mathcal{L}O(A,n)$ and $I = \cap_{i=1}^m I_i$ be a decomposition, where $V(I_i) \subseteq \mathrm{Spec}\,(A)$ are connected, as above, denote $\varepsilon_0(I,\omega_I) = \sum_{i=1}^m (I_i,[\omega_{I_i}]) \in \mathbb{Z}(\mathcal{L}O_0^c(A,n))$. Let $\mathscr{R}_0^n(A)$ denote the subgroup of $\mathbb{Z}(\mathcal{L}O_0^c(A,n))$, generated by the set $\{\varepsilon_0(I,\omega_I) : (I,\omega_I)$ is global and $height(I) \geq n\}$. In [BS2], the Euler class group $\mathcal{E}^n(A)$ was defined as

$$\mathcal{E}^n(A) = \frac{\mathbb{Z}(\mathcal{L}O_0^c(A,n))}{\mathscr{R}_0^n(A)}.$$

However, the differences between $E^n(A)$ and $\mathcal{E}^n(A)$ is superfluous, as follows.

Lemma A.2. *Use notations as in (A.1). The map $\varphi : \mathbb{Z}(\mathcal{L}O^c(A,n)) \longrightarrow \mathbb{Z}(\mathcal{L}O_0^c(A,n))$ sending $(I,\omega_I) \mapsto (I,[\omega_I])$ induces a surjective homomorphism $\Phi_n : E^n(A) \twoheadrightarrow \mathcal{E}^n(A)$. If $2n \geq d+3$, the Φ_n is an isomorphism.*

Proof. It is obvious that Φ_n is a well defined surjective homomorphism. To prove injectivity, let $\Phi_n(x) = 0$. Since, $2n \geq d+1$, by Moving Lemma 2.4 $x = \overline{\varepsilon}(I,\omega_I) \in E^n(A)$, for some $(I,\omega_I) \in \mathcal{L}O(A,n)$, with $height(I) \geq n$. It follows, $image(\varepsilon_0(I,\omega_I)) = 0 \in \mathcal{E}^n(A)$. By [BS2, Theorem 4.2], it follows ω_I lifts to a surjective map $A^n \twoheadrightarrow I$. Therefore, (I,ω_I) is global and $x = \overline{\varepsilon}(I,\omega_I) = 0$. Hence, Φ_n is an isomorphism. □

B. Elements of elementary orthogonal subgroups

This section is included due to nonavailability of suitable references. In this section, for a commutative ring A, the elements of A^n would be considered as column matrices. We consider the Orthogonal groups $O(A^{2n+1},q_{2n+1}) \subseteq GL_{2n+1}(A)$ and $O(A^{2n},q_{2n}) \subseteq GL_{2n}(A)$. In analogy to Elementary Subgroups of $GL_n(A)$, Elementary Orthogonal subgroups $\mathscr{E}O(A^{2n+1},q_{2n+1})$ and $\mathscr{E}O(A^{2n},q_{2n})$ of the respective Orthogonal groups were defined, which is classical (see [CF]).

For $i,j = 0,1,\ldots,2n, i \neq j$ and $\lambda \in A$, let $E_{ij}(\lambda) \in GL_{2n+1}(A)$ denote the matrix, whose diagonal entries are 1 and only other nonzero entry is λ at the $(i,j)^{th}$ position. Recall, the Elementary subgroup $EL_{2n+1}(A) \subseteq GL_{2n+1}(A)$ is generated by the set

$$\{E_{ij}(\lambda) : i,j = 0,1,\ldots,2n;\ i \neq j; \lambda \in A\}$$

For $i,= 0,1,\ldots,2n$, $j = 1,\ldots,2n$, $i \neq j$ and $\lambda \in A$, one can write down an orthogonal matrices $\varepsilon_{ij}(\lambda) \in O(A^{2n+1},q_{2n+1})$, by modifying $E_{ij}(\lambda)$, in a natural way. The elementary orthogonal subgroup $\mathscr{E}O(A^{2n+1},q_{2n+1})$ of $O(A^{2n+1},q_{2n+1})$ would generated by the set

$$\{\varepsilon_{ij}(\lambda) : i = 0,1,\ldots,2n; j = 1,\ldots,2n;\ i \neq j; \lambda \in A\},$$

while some $\varepsilon_{ij}(\lambda)$ would be redundant. A more precise definitions of $\varepsilon_{ij}(\lambda)$ are as follows.

Definition B.1. *Suppose A is a commutative ring and $n \geq 1$ is an integer. For our purpose, elements of A^{2n+1} would be dented by the columns*

$$\mathbf{u} := (z; x_1, \ldots, x_n; y_1, \ldots, y_n)^t.$$

For convenience, these $2n+1$ coordinates would be referred to as $0^{th}, 1^{st}, 2^{nd}, \ldots, n^{th}, \ldots, (2n)^{th}$-coordinates.

1. *For $j = 1, \ldots, n$, let $\varepsilon_{0,j}(\lambda)$ be the matrix that sends*

$$\mathbf{u} \mapsto (z + \lambda x_i; x_1, \ldots, x_n; y_1, \ldots, y_{i-1}, y_i - 2\lambda z - \lambda^2 x_i, y_{i+1}, \ldots, y_n)^t$$

For example, with $n = 2$, and $j = 1$, we have

$$\varepsilon_{0,1}(\lambda) = \begin{pmatrix} 1 & \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2\lambda & -\lambda^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2. *For $j = n+1, \ldots, 2n$, let $\varepsilon_{0,j}(\lambda)$ be the matrix that sends*

$$\mathbf{u} \mapsto (z + \lambda y_i; x_1, \ldots, x_{i-1}, x_i - 2\lambda z - \lambda^2 y_i, x_{i+1}, \ldots, x_n; y_1, \ldots, y_n)^t$$

For example, with $n = 2$, and $j = n + 1 = 3$, we have

$$\varepsilon_{0,3}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & \lambda & 0 \\ -2\lambda & 1 & 0 & -\lambda^2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

3. *For $i, j = 1, \ldots, n$, $i \neq j$, let $\varepsilon_{i,j}(\lambda)$ be the matrix that sends*

$$\mathbf{u} \mapsto (z; x_1, \ldots, , x_{i-1}, x_i + \lambda x_j, x_{i+1}, \ldots, x_n; y_1, \ldots, y_{j-1}, y_j - \lambda y_i, y_{j+1}, \ldots, y_n)^t$$

For example, with $n = 2$, and $i = 1$, $j = 2$, we have

$$\varepsilon_{1,2}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\lambda & 1 \end{pmatrix}$$

4. *For $i = 1, \ldots, n$; $j = n+1, \ldots, 2n$; $i \neq j - n$ (without loss of generality $i < j - n$), let $\varepsilon_{i,j}(\lambda)$ be the matrix that sends*

$$\mathbf{u} \mapsto (z; x_1, \ldots, x_i + \lambda y_j, \ldots, x_{j-n} - \lambda y_i, \ldots, x_n; y_1, \ldots, y_n)^t$$

For example, with $n = 2$, and $i = 1$, $j = n + 2 = 4$, we have

$$\varepsilon_{1,4}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \lambda \\ 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

5. *For $i = n+1, \dots, 2n$; $j = 1, \dots, n$; $i - n \neq j$ (without loss of generality $i - n < j$), let $\varepsilon_{i,j}(\lambda)$ be the matrix that sends*

$$\mathbf{u} \mapsto (z; x_1, \dots, x_n; y_1, \dots, y_{i-n} + \lambda x_j, \dots, y_j - \lambda x_{i-n}, \dots, y_n)^t$$

For example, with $n = 2$, and $i = n+1 = 3$, $j = 2$, we have

$$\varepsilon_{3,2}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & -\lambda & 0 & 0 & 1 \end{pmatrix}$$

Now we are ready to give the sedition of the Elementary Orthogonal subgroup $\mathscr{E}O(A, q_{2n+1}) \subseteq O(A, q_{2n+1})$.

Definition B.2. *Suppose A is a commutative ring. Then, the Elementary Orthogonal subgroup $\mathscr{E}O(A, q_{2n+1})$ of the Orthogonal group $O(A, q_{2n+1})$ is defined to be the subgroup, generated by the set of all $\varepsilon_{i,j}(\lambda)$ described and listed above* (B.1).

Likewise, the Elementary Orthogonal subgroup $\mathscr{E}O(A, q_{2n})$ of the Orthogonal group $O(A, q_{2n})$ is defined.

We give a proof of the following well known lemma, which we use in the proof of Lemma 4.1.

Lemma B.3. *Suppose $(A, \mathfrak{m})$ is a local commutative ring, with $1/2 \in A$. Let*

$$\mathbf{u} := (z; x_1, \dots, x_n; y_1, \dots, y_n)^t \in Q'_{2n}(A) \quad \text{and} \quad \mathbf{u}_0 := (1; 0, \dots, 0; 0, \dots, 0)^t.$$

Then, there is a matrix $\sigma \in \mathscr{E}O(A, q_{2n+1})$ such that $\sigma\mathbf{u} = \mathbf{u}_0$.

Proof. We prove this in several step.

Step 1. We prove that there is a matrix $\tau \in \mathscr{E}O(A, q_{2n+1})$, such that $\tau\mathbf{u} = (s;\ a_1, \dots, a_n;\ b_1, \dots, b_n)^t$, with a_1 unit.

If x_1 is a unit, there is nothing to prove. So, assume $x_1 \in \mathfrak{m}$. We have $z^2 + \sum_{i=1}^n x_i y_i = 1$. Since $x_1 \in \mathfrak{m}$, at least one of $z, x_2y_2, \dots, x_ny_n$ would be a unit.

1. Suppose y_j is unit, for some $j = 2, \dots, n$. Without loss go generality, assume y_2 is a unit. Then, (see Type 4) $\tau := \varepsilon_{1,n+2}(1)$ has the desired property.
2. Now, assume $y_j \in \mathfrak{m}$, $\forall\ j = 2, \dots, n$ and y_1 is a unit. Then, (see Type 3) $\varepsilon_{1,2}(1)(\mathbf{u}) = (z, x_1 + x_2, x_2, \dots, x_n;\ y_1, y_2 - y_1, y_3, \dots, y_n)^t$. Then, $y'_2 := y_2 - y_1$ is a unit. If $x'_1 := x_1 + x_2$ is a unit, then $\tau := \varepsilon_{1,2}(1)$ would have the desired property. If $x'_1 \in \mathfrak{m}$ then, $\tau = \varepsilon_{1,n+2}(1)\varepsilon_{1,2}(1)$ would have the desired property.
3. So, we assume $y_j \in \mathfrak{m}$, $\forall\ j = 1, \dots, n$ and hence z is a unit. Then, $\varepsilon_{0,n+1}(1)(\mathbf{u}) = (z + y_1;\ x_1 - 2z - y_1; x_2, \dots, x_n;\ y_1, \dots, y_n)$. Note, $x'_1 := x_1 - 2z - y_1$ is unit. So, $\tau = \varepsilon_{0,n+1}(1)$ has the desired property.

This completes the proof of Step 1.

Step 2. We prove that there is a matrix $\tau \in \mathscr{E}O(A, q_{2n+1})$, such that $\tau\mathbf{u} = (1;\ a_1, \dots, a_n;\ b_1, \dots, b_n)^t$, with a_1 unit.

To see this note that, by Step 1, we can assume that x_1 is a unit. Now, with $\lambda = x_1^{-1}(1 - z)$, let $\tau := \varepsilon_{0,1}(\lambda)$. Then, $\tau(\mathbf{u} = (1, x_1, \dots, x_n;\ y_1 - 2z\lambda - \lambda^2 x_1, y_2, \dots, y_n)^t$. So, τ has the desired property of Step 2.

Step 3. We prove that there is a matrix $\tau \in \mathscr{E}O(A, q_{2n+1})$, such that $\tau\mathbf{u} = (1;\ a_1, 0, \dots, \dots, 0;\ b_1, \dots, b_n)^t$, with a_1 unit.

By Step 3, we can assume $\mathbf{u} = (1, x_1, x_2, \dots, x_n;\ y_1, \dots, y_n)^t$, with x_1 unit. For $i = 2, \dots, n$, write (see Type 3) $\tau_i = \varepsilon_{1,i}(-x_1^{-1}x_i)$ and let $\tau = \tau_n\tau_{n-1}\cdots\tau_2$. Then $\tau(\mathbf{u}) = (1, x_1, 0, \dots, 0;\ y'_1, y_2, \dots, y_n)^t$ for some y'_1. This established Step 3.

Step 4. We prove that there is a matrix $\tau \in \mathscr{E}O(A, q_{2n+1})$, such that $\tau\mathbf{u} = (1;\ a_1, 0, \ldots, \ldots, 0;\ 0, \ldots, 0)^t$, with a_1 unit.

By Step 3, we can assume $\mathbf{u} = (1,\ x_1, 0, \ldots, 0;\ y_1, y_2, \ldots, y_n)^t$, with x_1 unit. Since $\mathbf{u} \in Q'_{2n}(A)$, we have $1 + x_1y_1 = 1$ and hence $y_1 = 0$. Therefore, $\mathbf{u} = (1,\ x_1, 0, \ldots, 0;\ 0, y_2, \ldots, y_n)^t$, with x_1 unit. For $i = 2, \ldots, n$, write (see Type 5) $\tau_i = \varepsilon_{n+i,1}(-x_1^{-1}y_i)$ and let $\tau = \tau_n\tau_{n-1}\cdots\tau_2$.

Since $x_2 = x_3 = \cdots = x_n = 0$ and $y_1 = 0$, we have $\tau(\mathbf{u}) = (1, x_1, 0, \ldots, 0;\ 0, 0, \ldots, 0)^t$, with x_1 unit. This establishes Step 4.

The Final Step. By Step 4, we can assume $\mathbf{u} = (1; x_1, 0, \ldots, \ldots, 0;\ 0, \ldots, 0)^t$, with x_1 unit. Now, let $\tau = \varepsilon_{0,n+1}(\frac{x_1}{2})$. Then, $\tau(\mathbf{u}) = (1; 0, \ldots, 0;\ 0, \ldots, 0)$. The proof is complete. □

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